

GEOMETRIC TOMOGRAPHY, SECOND EDITION  
CORRECTIONS AND UPDATE  
Version 1.8, October 30, 2024

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1. PREFACE

Many years have passed since the publication of the second edition of *Geometric Tomography* and much has happened. One major event was the appearance in 2014 of the second edition of Schneider's classic text [216], which for some time lessened the pressure to post an update to my own book. But such has been the pace of advances in convex geometry and related areas that it has finally become necessary to take action.

To keep the task manageable, the update is limited, at least for now, to a list of corrections and a set of reports on the problems stated at the end of the chapters. The idea is to post new versions of this manuscript, each numbered and dated, and widen the scope as time and energy allow, for example by adding reports on other open problems in geometric tomography.

A brief overview of progress on the problems can be found at the beginning of Section 4. There are many, particularly from Chapters 1, 2, 5, and 6, for which there is little or nothing to say. In fact, the problems vary widely in perceived importance. At one extreme stand the slicing problem, Problem 8.3, and Mahler's conjecture, Problem 9.2, whose significance and notoriety, not just in convex geometry but in mathematics as a whole, continue to increase steadily. At the other, there are problems that arose in the course of writing the first edition of the book and seemed potentially accessible to undergraduate and Masters students. These were motivated by the study of X-rays of convex and star bodies, a topic that although not the focus nowadays of much attention, does lend itself to natural problems, some of which may well await beautiful solutions. For example, Problem 2.1 is still untouched, despite appearing as Question 1 in my 1995 *AMS Notices* article on geometric tomography.

The two problems singled out above, the slicing problem and Mahler's conjecture, present by far the greatest challenge in adequately summarizing their status. This is due to the invasion of methods, results, and conjectures from other areas of geometry, such as contact geometry and symplectic geometry, and from reaches of analysis beyond those encountered in the book. Hundreds of pages have already been devoted to them in expository works, several of which are masterly and go into far more depth than the short reports presented here. Nevertheless, there does not appear to be any single source in the literature that provides a comprehensive overview of either problem. Figure 1 on page 13 raises questions which, while obvious, have not appeared in the literature. For example, is KLS equivalent to its centered (i.e., origin-symmetric) version, thereby eliminating the need for the two dotted arrows? Are

there any other relations between the 40 conjectures in the diagram? In particular, is it true that  $\text{SISO2} \Rightarrow \text{MAH2}$ , a relation that would connect the graph?

Rolf Schneider's eagle eye spotted a number of typos and this help is much appreciated. I am also very grateful to the following for helpful correspondence related to the specified aspects of this update: Juan Carlos Álvarez Paiva (Problem 9.2), Florent Balacheff (Problem 9.2), Paolo Dulio (Problems 1.5 and 5.6), Apostolos Giannopoulos (Problem 8.3), David Jerison (Problem 8.3), Roman Karasev (Problem 9.2), Bo'az Klartag (Problems 8.3 and 9.2), Alex Koldobsky (Problems 7.6 and 8.3), Emanuel Milman (Problems 8.3 and 9.3 to 9.5), Luis Montejano (Problems 3.3 and 7.4), Boris Rubin (Problem 8.5), Dmitry Ryabogin (Problems 3.1, 3.2, 7.2, and 7.3), Grzegorz Sójka (correction for p. 52), Yanir Rubinstein (Problems 8.3 and 9.2), Gaoyong Zhang (Problems 4.4, 8.3, 8.6, 8.7, 9.2, and 9.3), and Artem Zvavitch (Problem 9.2).

Any further mistakes or misprints, as well as news about the open problems, can be sent to [Richard.Gardner@wwu.edu](mailto:Richard.Gardner@wwu.edu). A current version of this document will be kept at <http://faculty.wwu.edu/gardner/> (click on Research).

## 2. REMARKS ON TERMINOLOGY, NOTATION, AND NAMES

In preparing the second edition, I felt some obligation to purchasers and readers of the first edition not to make too many changes in basic terminology and notation. However, even then my taste had moved with the times, and since its appearance in 2014, the second edition of Rolf Schneider's classic text also has to be taken into account. Regarding notation, nowadays I prefer:

1.  $\mathbb{R}^n$  rather than  $\mathbb{E}^n$  for  $n$ -dimensional Euclidean space.
2.  $B^n$  exclusively (and not  $B$ ) for the unit ball in  $\mathbb{R}^n$ .
3.  $\partial E$  for the boundary of  $E$  instead of  $\text{bd } E$ .
4.  $\mathcal{K}_n^n$  instead of  $\mathcal{K}_0^n$  for the class of convex bodies in  $\mathbb{R}^n$ .
5.  $\mathcal{H}^n$  rather than  $\lambda_n$  for  $n$ -dimensional Lebesgue or Hausdorff measure, and  $V$  or  $V_n$  when the set in question is a body.
6.  $K^\circ$  instead of  $K^*$  for the polar body of  $K$ , though the latter is consistent with the notation for the dual space in functional analysis.

In terminology, I now favor:

1. *o-symmetric* or *origin symmetric* rather than *centered* (see p. 3 of the book). Unfortunately, "centered" is sometimes used to mean having centroid at the origin. (The term *centrally symmetric* seems increasingly to be abused; its meaning is clear and is not the same as origin symmetric.)
2. *dilation* instead of *dilatation* (see p. 5 of the book), as almost everyone else does, even though technically the latter is correct.

Despite these current preferences, notation and terminology in what follows will generally adhere to that in the book, and the meaning of symbols and terms can be found there.

A word about names. Several family names, such as Milman and Zhang, are shared by more than one individual in the community. Nevertheless, only the last name is given when citing an author in the text, since the initials are available in the bibliography.

### 3. CORRECTIONS AND AMENDMENTS

Page	Line	Comments
52	1–2	It has come to my attention that “all points in $V$ are isolated” does not contradict Lemma 1.2.26, as claimed. Thus the proof of Lemma 1.2.27, if the statement is true, is unfortunately incomplete.
54	-2	Replace “(a)” by “(b)”.
110	14	The proof of Theorem 3.2.7 should start as follows: We may assume without loss of generality that the center of $K$ is at the origin. Then origin symmetry implies that $h_K(u) = h_K(-u)$ , and...
126	-12	Replace “constants $a$ and $b$ ” by “ $a, b > 0$ ”.
128	-15	Insert “and $k = n - 1$ ” after “odd”.
133	13	The small indent at the left should be removed.
133	-2	Replace “ $i = 1, j = 2$ ” by “ $i = 1, j = k = 2, n = 3$ ”.
134	2–9	The results of Howard and Hug were actually obtained under the (formally) stronger assumption that the $i$ th and $j$ th projection functions of $K_1$ and $K_2$ are proportional, in other words, $V_i(K_1 S) = a V_i(K_2 S)$ , for all $S \in \mathcal{G}(n, i)$ , and $V_j(K_1 T) = b V_j(K_2 T)$ , for all $T \in \mathcal{G}(n, j)$ .
185	-12	The fact that the unit cube in $\mathbb{E}^n$ has property (VP) was proved independently by Chakerian and Filliman [55] and attributed by them to Michael Kallay.
204	6–7	Delete “, if $i \geq 0$ , and $\nu_k(E_1) > \nu_k(E_2)$ , if $i < 0$ ”.
205	13–15	Replace this sentence by “If $i < 0$ , the same conclusion is reached after noting that in this case $s_j(\theta)^i \leq r_j(\theta)^i$ , for $j = 1, 2$ , and there is an extra change of sign in the integrand.”
213	Fig. 5.9	The bold and dotted lines bounding the component $C$ should be interchanged. As it is, both $C$ and $p_3C$ are in $K \setminus K'$ .
226	20	Insert before the last sentence in the first paragraph of Note 5.1: “When $p = o$ and $E$ is a bounded domain containing a ball with center $o$ , the directed chordal symmetral $D_p E$ of a set $E$ at a point $p$ was first defined by Bandle and Marcus [27], who called it the <i>radial concentration</i> of $E$ and utilized it in the theory of capacities. Their definition was actually more general, allowing the use of some non-Euclidean metrics.”
341	22–23	Replace “Problems 8.6 and 8.7” by “Problem 8.6” and replace “the hypotheses then imply” by “when $n \geq 3$ , the stronger hypothesis $IL = cL$ for some $c > 0$ implies”.

Page	Line	Comments
345	19–24	The results quoted on these lines only pertain to the special case of Theorem 8.2.13 when $L$ is also a centered convex body. (For an update on the slicing problem, see below.)
373	-7	It might be noted that the cases $i = n - 1$ and $i = 1$ of Corollary 9.4.8 are consequences of the equality conditions in Theorem 9.3.1 (right-hand inequality) and Theorem 9.3.2, respectively.
381	-8	Replace “case $p = 1$ was” by “cases $p = 1$ and $p = 2$ were”.
385	-12	Insert “, the centroid of $M$ is at the origin,” after “ $\lambda_n(M) = 1$ ”.

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#### 4. REPORTS ON THE PROBLEMS

Citations such as [153book] refer to the references in the book, and numbers associated to pages, equations, theorems, or notes without reference to another work refer to those in the book. A brief overview is as follows, where for problems not listed, any progress is detailed in the full reports below.

**Solved:** Problems 8.8, 8.9(i), 9.3.

**Solved but with modified versions open:** Problems 3.1, 3.2, 7.2, 7.3.

**Solved in significant special cases:** Problem 3.3 (for arbitrary  $n$  and  $k \equiv 1 \pmod{4}$ ,  $k \neq 133$ ), Problem 3.9 (assuming that  $K_1$  and  $K_2$  have proportional  $i$ th and  $j$ th projection functions,  $1 \leq i < j \leq n - 2$ ,  $(i, j) \neq (1, n - 2)$ , and  $K_2$  enjoys a certain weak regularity condition), Problem 7.4 (when  $L$  is a convex body containing  $o$ ,  $n$  is arbitrary, and  $i \equiv 1 \pmod{4}$ ,  $i \neq 133$ ), Problem 7.6 (when  $L_1$  and  $L_2$  are polytopes), Problems 8.6 and 8.7, when  $i = n - 1$  and “homothetic to” is replaced by “a dilatate of,” Problem 9.2 (in its centered version, when  $n = 3$ ), Problem 9.4 (when  $i = 1$ ).

**Problem 1.1**

The problem is also stated in [68]. One may as well take the two X-rays in the coordinate directions. Vincze and Nagy [222] define a map  $\Phi$  that takes a planar compact convex set  $K$  to its *generalized conic function*  $f_K(x)$ , defined as the integrated taxicab distance of  $x$  to the points in  $K$ . Up to a constant, the second partial derivatives of  $f_K$  are the X-rays of  $K$  in the coordinate directions. In [222, Theorem 5], it is shown that  $K$  is determined by its X-rays in the coordinate directions if and only if the set-valued inverse of  $\Phi$  is lower semi-continuous at  $f_K$ , and in [222, Remark 3] the authors speculate on how this condition might be utilized.

**Problem 1.5**

The problem remains unsolved, but there are results dealing with the case when  $\varepsilon = 0$ . Dulio, Longinetti, Peri, and Venturi [69, Theorem 1.1] prove that if  $K, L \in \mathcal{K}_0^2$  and  $X_{u_i}K = X_{u_i}L$  for  $i = 1, \dots, m$ ,  $m \geq 3$ , then

$$\lambda_2(K \Delta L) \leq \frac{1 - \cos(\pi/m)}{\cos(\pi/m)} \lambda_2(K \cap L),$$

with equality if and only if, up to a nonsingular affine transformation,  $K$  and  $L$  are regular  $m$ -gons,  $K$  is a rotation of  $L$  about  $o$  by  $\pi/m$ , and  $\{u_1, \dots, u_m\}$  are the corresponding equally spaced directions. Some refinements and variations are also given in [69], including estimates in terms of cross ratios, when  $m \geq 4$ , as well as when the directions are only known up to a fixed error.

**Problem 2.2**

The problem is also stated in [68].

**Problem 2.8**

The problem in the form stated is still open. However, it is not true that if  $\mathcal{S}$  is a finite set of directions in  $\mathbb{E}^2$  such that planar convex bodies are determined by their X-rays in the directions in  $\mathcal{S}$ , then planar convex bodies are also determined among measurable sets by their X-rays in the directions in  $\mathcal{S}$ . Indeed, let  $\mathcal{S} = \{(1, 0), (0, 1), (2, 1), (-1, 2)\}$ . It was proved by Gardner and Gritzmann [267book, Theorem 6.2(i)] that planar convex bodies are determined by their X-rays in the directions in  $\mathcal{S}$ . We claim that if  $K = \text{conv}\{(2, 4), (-2, -4), (4, -2), (-4, 2)\}$ , a parallelogram, then there are an  $r > 1$  and a measurable set  $E$  in  $\mathbb{E}^2$  essentially different from  $rK$  such that  $rK$  and  $E$  have the same X-rays in the directions in  $\mathcal{S}$ . To see this, we use an  $\mathcal{S}$ -switching component discovered by Gritzmann, Langfeld, and Wiegmann [95]. In [95, Fig. 1, p. 1594], two finite lattice sets are depicted. Denoting the left set by  $A$  and the right set by  $B$ , we see that  $A$  consists of the integer lattice points contained in the parallelogram  $K$ . If  $F = A \setminus B$  and  $G = B \setminus A$ , then  $|F| = |G| = 8$  and  $F \cup G$  is an  $\mathcal{S}$ -switching component. Moreover,  $F \subset \text{bd}K$  and  $G \subset \mathbb{E}^2 \setminus K$ . Since  $G$  is finite, we may choose  $r > 1$  such that  $G \subset \mathbb{E}^2 \setminus rK$  and of course  $F \subset \text{int}rK$ . Now let  $\varepsilon > 0$  be small enough that  $F + \varepsilon B^2 \subset \text{int}rK$  and  $G + \varepsilon B^2 \subset \mathbb{E}^2 \setminus rK$ . Setting  $E = (rK \setminus (F + \varepsilon B^2)) \cup (G + \varepsilon B^2)$ , it is clear that  $E$  is a measurable set with the same X-rays as  $rK$  in the directions in  $\mathcal{S}$ .

**Problem 3.1**

Note the restriction  $2 < k < n - 1$ , which eliminates  $\mathbb{E}^3$  as a setting for this problem. The reason is that the Petty–McKinney example in Theorem 3.1.8 already gives a negative answer when the projections are 2-dimensional. In any case, the examples constructed by Zhang [227, Theorem 1.4] answer this problem negatively, even when the dilatation factor for each projection is one. See the report for Problem 3.2 for more details.

It seems that the following modified version of Problem 3.1, inspired by [227, Remark 4.2], is open.

*Suppose that  $2 < k \leq n - 1$ , and that  $K_1$  and  $K_2$  are convex bodies in  $\mathbb{E}^n$  such that  $K_1|S$  is similar to  $K_2|S$  for all  $S \in \mathcal{G}(n, k)$ . Is  $K_1$  equal to  $K_2$  up to a dilatation, a translation, and a reflection in a proper subspace?*

**Problem 3.2**

Zhang [227, Theorem 1.4] solves this problem negatively. However, the bodies he constructs are reflections of each other in a nontrivial proper subspace.

The following positive partial results have been obtained.

Myroshnychenko and Ryabogin [181] show that the answer is affirmative if one of the bodies is a polytope.

Ryabogin [206] proves that if  $f, g \in C(S^{n-1})$ ,  $n \geq 3$ , and for each  $S \in \mathcal{G}(n, 2)$ , there is a rotation  $\phi_S$  in  $S$  about  $o$  such that  $f(\phi_S u) = g(u)$  for each  $u \in S^{n-1} \cap S$ , then  $f(u) = g(\pm u)$  for each  $u \in S^{n-1}$ . As a corollary, he concludes that if  $K|S$  is a rotation in  $S$  about  $o$  of  $L|S$  for each  $S \in \mathcal{G}(n, 2)$ , then  $K = \pm L$ . The case  $n = 3$  of the latter result was also obtained by Mackey [161].

Extra conditions can be placed on  $K$  and  $L$  that ensure that the desired conclusion, or a slightly modified version of it, is true. These conditions stipulate the existence of diameters or forbid symmetries in the projections. The details are rather involved, so we refer the reader to Ryabogin [207], Alfonseca, Cordier, and Ryabogin [7], and the references given in these papers.

The following modified versions of Problem 3.2 remain open. The first was posed by Ryabogin (private communication) and the second is taken from [227, Remark 4.2].

*Suppose that  $K_1$  and  $K_2$  are convex bodies in  $\mathbb{E}^3$  such that  $K_1|S$  is directly congruent to  $K_2|S$ , for all  $S \in \mathcal{G}(3, 2)$ . Is  $K_1$  a translate of  $\pm K_2$ ?*

Here, *directly congruent* means equal up to a direct rigid motion.

*Suppose that  $2 < k \leq n - 1$ , and that  $K_1$  and  $K_2$  are convex bodies in  $\mathbb{E}^n$  such that  $K_1|S$  is congruent to  $K_2|S$  for all  $S \in \mathcal{G}(n, k)$ . Are  $K_1$  and  $K_2$  equal, up to a translation and a reflection in proper subspace?*

Ryabogin [208] surveys Problem 3.2 and poses other questions of a similar nature; answers to some of these have been found by Myroshnychenko, Ryabogin, and Saroglou [182] and Zhang [227, p. 2065].

**Problem 3.3**

Montejano [179] supplies an affirmative answer when  $n$  is arbitrary and  $k \equiv 1 \pmod{4}$ ,  $k \neq 133$ . This leaves open the cases when  $n$  is arbitrary and  $k = 133$  or  $k \equiv 3 \pmod{4}$ . In his proof, Montejano uses some of the results in [45], which solved the corresponding cases of Problem 7.4, the question for sections dual to Problem 3.3.

Montejano [180] provides a detailed commentary on [179] and also obtains an affirmative answer when  $k = n - 2$  is odd, under the stronger assumption that all the projections of  $K$  are linearly equivalent; see [180, Theorem 5.8].

See the report for Problem 7.4 for further comments.

**Problem 3.9**

Howard and Hug [401book], [402book] obtained partial solutions to Problem 3.9 under the (formally) stronger assumption that

$$V_i(K_1|S) = a V_i(K_2|S), \quad \forall S \in \mathcal{G}(n, i)$$

and

$$V_j(K_1|T) = b V_j(K_2|T), \quad \forall T \in \mathcal{G}(n, j),$$

in other words, the  $i$ th and  $j$ th projection functions of  $K_1$  and  $K_2$  are proportional. (It is stronger by Kubota's integral recursion (A.46), p. 408.) See p. 134 and the corrections listed above in this update. For convenience, if not historical accuracy, we shall refer to the special case when  $K_2 = B^n$  of this modified version of Problem 3.9 as *Nakajima's problem*. This includes the question as to whether a convex body of constant width and constant brightness must be a ball. Building on earlier work, Hug [109] obtains an affirmative answer to the same modified version of Problem 3.9, when  $1 \leq i < j \leq n - 2$ ,  $(i, j) \neq (1, n - 2)$ , and  $K_2$  enjoys a certain weak regularity condition (satisfied, for example, when  $\text{bd } K_2$  contains a small region in which it is  $C_+^2$ ). It follows that Nakajima's problem has an affirmative answer for these values of  $i$  and  $j$ , but remains open for arbitrary  $K_1$  when  $(i, j) = (1, n - 2)$  and  $n \geq 6$ , and when  $1 \leq i \leq n - 2$ ,  $j = n - 1$ , and  $n \geq 4$ . More is known when  $K_1 \in C^2$  but there has been no progress beyond what is already stated in Note 3.6.

**Problem 4.4**

Saroglou and Zvavitch [215] show that if  $n \geq 3$  and there is a  $\phi \in GL(n)$  such that the curvature function  $f_{\phi K}$  exists and is sufficiently close to 1 (i.e.,  $\|f_{\phi K} - 1\|_\infty \leq \varepsilon$  for some  $\varepsilon > 0$ ), then  $\Pi^m K \rightarrow B^n$  as  $m \rightarrow \infty$  in the Banach–Mazur metric. From this they are able to conclude that if  $K$  satisfies these conditions and  $\Pi^2 K$  is homothetic to  $K$ , then  $K$  is an ellipsoid. The latter result was obtained independently by Ivaki [115] with an extra smoothness assumption on  $K$ .

It is already mentioned in Note 4.6 that when  $i = 1$ , Schneider [732book] proves that  $K$  must be a ball. Ivaki [114] addresses Problem 4.4 for the case when  $1 < i < n - 1$  and shows that with the same assumptions on  $K$  as in [115], if  $\Pi_i^2 K$  is homothetic to  $K$ , then  $K$  is a ball.



Ivaki's results mentioned above are generalized to sufficiently regular even Minkowski valuations by Ortega-Moreno and Schuster [188].

Though they do not bear directly on Problem 4.4, related results of Saraglou [214] are worth mentioning. He proves that if  $K \in \mathcal{K}_0^3$ , then  $\Pi^2 K \subset 8\lambda_3(K)K$  if  $K$  is a zonoid and  $6\lambda_3(K)K \subset \Pi^2 K$  if  $K$  is centrally symmetric, both inclusions being sharp.

### ***Problem 5.6***

The problem as stated is completely open, but Dulio, Longinetti, Peri, and Venturi [69, Section 6] apply their results on Problem 1.5, together with Theorem 6.2.8, to obtain stability estimates for planar convex bodies with equal  $-1$ -chord functions at finitely many points.

### ***Problem 7.2***

This is the dual version of Problem 3.1. The examples constructed by Zhang [227, Theorem 1.4] provide a negative answer. See the reports for Problems 3.1, 3.2, and 7.3 for more details.

### ***Problem 7.3***

This is the dual version of Problem 3.2. All the results, both positive and negative, reported for Problem 3.2 have corresponding versions for Problem 7.3 and are stated in the articles referred to for Problem 3.2. Suitably modified versions of Problem 7.3 may also be posed, in analogy to those for Problem 3.2; of course, translation is not permitted in this case.

### ***Problem 7.4***

This is the dual version of Problem 3.3. For convex  $L$  containing  $o$ , the problem is attributed to Banach [26] (see p. 244 of the French original or p. 152 of the English translation), where, of course, it is phrased in the language of Banach spaces.

Bor, Hernández Lamonedá, Jiménez-Desantiago, and Montejano [45] achieve a remarkable breakthrough by proving the following result.

*Problem 7.4 has an affirmative answer when  $L$  is a convex body containing  $o$ ,  $n$  is arbitrary, and  $i \equiv 1 \pmod{4}$ ,  $i \neq 133$ .*

In this setting (when  $L$  is convex containing  $o$ ), this leaves open the cases when  $i = n - 1$  and  $i = 133$  or  $i \equiv 3 \pmod{4}$ . Apparently the reason for the exclusion of  $i = 133$  stems from the fact that 133 is the dimension of the exceptional Lie group  $E_7$ ; see [45, Theorem 1.6]. The authors of [45] use the following new result in their proof.

*If  $K \in \mathcal{K}_0^n$ ,  $n \geq 3$ , is centered and such that all its sections by hyperplanes are linearly equivalent affine images of bodies of revolution in their respective hyperplanes, then  $K$  is an ellipsoid.*

In a long survey article with some new results and several proofs, Montejano [180] gives a detailed commentary on [45] and [179], and revisits and reworks Gromov's result and ideas in [359book] (see Note 7.2).

As far as I know, this problem is still completely open when  $L$  is a non-convex star body in  $\mathbb{E}^n$ .

**Problem 7.6**

Howard, Nazarov, Ryabogin, and Zvavitch [107] answer the question affirmatively for star bodies of revolution in  $\mathbb{E}^3$  with  $C^1$  radial functions. For the more general problem in  $\mathbb{E}^n$ , Yaskin [226] obtains an affirmative answer when  $L_1$  and  $L_2$  are convex polytopes.

Ryabogin and Yaskin [209, Theorem 2.4] show that the assumption that  $L_1$  and  $L_2$  are centered is necessary. Their result is as follows.

*There are noncongruent convex bodies  $L_1$  and  $L_2$  in  $\mathbb{E}^n$ , containing the origin in their interiors, such that for  $1 \leq i \leq k \leq n - 1$ , we have*

$$V_i(L_1 \cap S) = V_i(L_2 \cap S),$$

*for all  $S \in \mathcal{G}(n, k)$ . Moreover,  $L_1$  and  $L_2$  can be constructed in such a way that both are  $C_+^\infty$  bodies of revolution or both are polytopes.*

Note that the same result with  $V_i$  replaced by  $\tilde{V}_i$  is implied by Theorems 7.2.13 and 7.2.14. The problem as stated, in all its versions, is still open.

Related to Problem 7.6 is the following Busemann-Petty problem for surface areas.

*Let  $K_1$  and  $K_2$  be centered convex bodies in  $\mathbb{E}^n$  such that*

$$\lambda_{n-2}((\text{bd } K_1) \cap u^\perp) \leq \lambda_{n-2}((\text{bd } K_2) \cap u^\perp),$$

*for all  $u \in S^{n-1}$ . Is it true that  $\lambda_{n-1}(\text{bd } K_1) \leq \lambda_{n-1}(\text{bd } K_2)$ ?*

The question was posed by König and Koldobsky [152], who showed that the answer is negative if  $K_1$  is a unit cube and  $K_2$  is a ball of suitable radius, provided  $n \geq 14$ . Brazitikos and Liakopoulos [50] consider the isomorphic version of this question, i.e., whether the hypotheses imply that  $\lambda_{n-1}(\text{bd } K_1) \leq C\lambda_{n-1}(\text{bd } K_2)$  for a universal constant  $C$ , along with versions of the slicing problem for surface area.

**Problem 8.2**

Rubin [204] shows that the answer is affirmative when the body with smaller sections is invariant under rotations preserving a pair of mutually orthogonal subspaces of dimensions  $k$  and  $n - k$ ,  $1 \leq k \leq n - 1$  (the case  $k = 1$  corresponding to a body of revolution), satisfying  $i \leq \min\{k, n - k\}$ . The problem remains open.

**Problem 8.3**

The slicing problem, which might be regarded as the Holy Grail of convex geometry, has so far resisted all attacks, despite many amazing contributions and advances in understanding its relation to other problems. Schneider [216, pp. 605–6] presents a very brief summary of developments prior to 2014. Here we shall refer both to the slicing problem and the *hyperplane conjecture* that the answer to the problem is affirmative. The term *hyperplane problem* is also in use, but has the disadvantage of also applying to Banach's question as to whether every infinite-dimensional Banach space is isomorphic to its hyperplanes. It will be convenient to formulate several other related problems as conjectures, without implying that anyone strongly believes them to be true.

Milman and Pajor's paper [621book], already cited in Note 9.8, was the first comprehensive study of the topic. In addition to the lecture notes of Giannopoulos [89], there are several other recent expositions, ranging from the pithy and timely advertisement by Klartag and Werner [140] to long chapters or whole books, by Alonso-Gutiérrez and Bastero [9], Artstein-Avidan, Giannopoulos, and Milman [16, Chapter 10], Brazitikos, Giannopoulos, Valettas, and Vritsiou [49], and Klartag and Milman [139]. It transpires that the slicing problem is related to several other major open problems, and we draw on all these sources and others in an attempt to describe succinctly the current state of the art. Figure 1 (compare [157, Fig. 2]) shows the known relations between the various conjectures, but at the top levels some basic questions are unresolved, or have answers not yet published and fully understood only by a small group of experts.

The slicing problem or hyperplane conjecture originated with the work of Bourgain [46], motivated by harmonic analysis; see the remark after [46, Lemma 2], as well as [16, pp. 360–1], [49, p. 141], and [139] for more details. In the context of centered convex bodies, some basic information, including the equivalence of the hyperplane conjecture to the *isotropic constant conjecture* (ISO) that  $L_K \leq C$  for any isotropic centered convex body in  $\mathbb{E}^n$  and a universal constant  $C$  (see (9.11), p. 385), is already set out in Note 9.8. However, both conjectures have versions that are apparently more general but turn out to be equivalent. Paouris [190] proved that we could also state the hyperplane conjecture without the symmetry assumption, as follows.

*Hyperplane conjecture* (HYP). There is a universal constant  $C > 0$  such that if  $K \in \mathcal{K}_0^n$  and  $\lambda_n(K) = 1$ , there is a hyperplane  $H$  such that  $\lambda_{n-1}(K \cap H) > C$ .

(Here, and in what follows, possibly different universal constants are all denoted by  $C$ .) For ISO, we first need to extend some basic concepts. An arbitrary  $K \in \mathcal{K}_0^n$  is *isotropic* if it satisfies the conditions on p. 385 and in addition has its centroid at the origin. For isotropic convex bodies,  $L_K$  is defined as for centered bodies, and one observes that any  $K \in \mathcal{K}_0^n$  has an isotropic image  $\phi K$  under some  $\phi \in GA_n$ . Its *isotropy constant*  $L_K$  can then be defined as the isotropy constant of this affine image. Explicit formulas for  $L_K$ , such as those in [216, p. 605] or (8) below, are available. With this understanding, ISO is equivalent to the statement that  $L_K \leq C$  for any convex body in  $\mathbb{E}^n$  and a universal constant  $C$ . This follows from Klartag's observation in [127, Remark 2, p. 392] (see also [16, Proposition 10.2.15], [49, Proposition 2.5.10], or [89, Proposition 8.9]) that there is a universal constant  $C$  such that for any isotropic convex body  $K$  in  $\mathbb{E}^n$ , there is a centered convex body  $K_0$  such that  $L_K \leq C L_{K_0}$ .

In fact, ISO may also equivalently be formulated for log-concave measures. An account of the intriguing interplay between convex bodies and log-concave measures can be found in [216, Section 9.5] (see also [16, Section 10.2], [49, Section 2.5], or [89, Sections 7 and 8]). The core result is that if  $f$  is a nonnegative log-concave function on  $\mathbb{E}^n$  with  $f(o) > 0$  and finite positive integral, and  $p > 0$ , then

$$(1) \quad K_p(f) = \left\{ x \in \mathbb{E}^n : p \int_0^\infty f(rx) r^{p-1} dr \geq f(o) \right\}$$

defines a convex body in  $\mathbb{E}^n$ . See, for example, [49, Proposition 2.5.5] or [284book, Corollary 4.2]; this is due to Ball, who stated it in [23, p. 74], [20book] for even  $f$  and  $p \geq 1$ . Moreover,  $K$  is centered if  $f$  is even, and if  $K$  is a convex body with  $o \in \text{int } K$ , then  $K_p(1_K) = K$ . In fact, Klartag's observation [127, Remark 2, p. 392], [49, Proposition 2.5.10] mentioned in the previous paragraph uses this construction, since one can take  $K_0 = K_{n+2}(f)$ , where  $f = g_K$  is the covariogram of  $K$  defined on p. 377, which is easily shown to be log-concave by the Brunn–Minkowski inequality, and satisfies  $g_K(o) = \lambda_n(K) > 0$ . See also [166], where Martín-Goñi finds for each  $n$  the optimal constant  $d_n$  such that  $L_K \leq d_n L_{K_{n+2}(g_K)}$ .

Here we take a *log-concave measure* to be a probability measure in  $\mathbb{E}^n$  with a log-concave density function  $f$ . Such a measure  $\mu$  is called *isotropic* if it has its centroid at the origin and

$$(2) \quad \int_{\mathbb{E}^n} (u \cdot x)^2 d\mu(x) = 1$$

for all  $u \in S^{n-1}$ , and its *isotropic constant* is defined by

$$L_\mu = \sup_{x \in \mathbb{E}^n} f(x)^{1/n}.$$

Then ISO, and therefore HYP, can be reinterpreted as asking whether there is a universal constant  $C$  such that  $L_\mu \leq C$  for all such  $\mu$ . This is because an extension of a result of Ball [20book] in [16, Proposition 10.2.16], [49, Proposition 2.5.12] or [89, Proposition 8.10] shows that if  $\mu$  is an isotropic log-concave measure in  $\mathbb{E}^n$  with density  $f$ , then there are universal constants  $c_1, c_2 > 0$  such that

$$c_1 L_\mu \leq L_{K_{n+1}(f)} \leq c_2 L_\mu,$$

where  $K_{n+1}(f)$  is the convex body defined by (1) with  $p = n + 1$ .

To summarize, in all their versions, we have

$$\text{HYP} \Leftrightarrow \text{ISO}.$$

(A proof of this equivalence is also given in [49, Theorem 3.1.2 and pp. 107–8], with the help of several results from [49, Section 2.2].)

Functional versions of ISO have been considered and are extensively studied by Fradelizi and Marín Sola [79]. If  $f$  is a nonnegative integrable log-concave function on  $\mathbb{E}^n$  with nonzero integral, one can define its isotropic constant by

$$(3) \quad L_f = \left( \frac{\max_{x \in \mathbb{E}^n} f(x)}{\int_{\mathbb{E}^n} f(x) dx} \right)^{1/n} (\det \text{Cov}(f))^{1/2n},$$

where  $\text{Cov}(f)$  is the covariance matrix of  $f$ . (When  $f$  is the density function of an isotropic log-concave measure,  $L_f$  equals the isotropic constant  $L_\mu$  of  $\mu$  defined above.) For reasons explained below after the introduction of the sharp isotropic constant conjectures, a preference is expressed in [79] for a different definition, namely

$$(4) \quad \widehat{L}_f = \left( \frac{e^{-h(f)}}{\int_{\mathbb{E}^n} f(x) dx} \right)^{1/n} (\det \text{Cov}(f))^{1/2n},$$

where  $h$  is the *differential entropy* of  $f$ , i.e.,

$$(5) \quad h(X) = - \left( \int_{\mathbb{E}^n} f(x) \log f(x) dx \right) / \int_{\mathbb{E}^n} f(x) dx.$$

Each definition of the isotropic constant of  $f$  gives rise to a functional version of ISO, namely that it is bounded above by a universal constant. It follows from [79, Remark 3.1] that each is equivalent to HYP.

Henceforth we shall give preference to HYP in stating its consequences, even when they follow more immediately from ISO.

The discussion in Note 9.8 mentions a couple of other statements equivalent or related to HYP, for example one due to Klartag and Milman [443book] involving Steiner symmetrization, to which should be added a result in a note of Ball [24]. Others still were already known to Milman and Pajor [621book]. In addition to some of these discussed below, there is a connection to the following stronger form of Milman's reverse Brunn–Minkowski inequality (see [16, Theorem 8.4.3] and [216, p. 380]).

*Reverse Brunn–Minkowski inequality conjecture* (RBM). There is a universal constant  $C > 0$  such that if  $K, L \in \mathcal{K}_0^n$  are isotropic, then

$$\lambda_n(K + L)^{1/n} \leq C (\lambda_n(K)^{1/n} + \lambda_n(L)^{1/n}).$$

In [621book, pp. 78–9] (see also [49, p. 111]), it is shown that HYP  $\Rightarrow$  RBM and credit is given to Ball for first proving this. (The reference in [621book] to Ball's 1986 PhD thesis does not appear to be correct.) Bourgain, Klartag, and Milman [47, Proposition 1.4] (see also [49, Theorem 3.2.4]) prove the reverse implication, so in fact

$$\text{HYP} \Leftrightarrow \text{RBM}.$$

Another statement from Milman and Pajor [621book] is phrased in the following slightly different form in [139].

*Ellipsoid intersection conjecture* (ELL). There is a universal constant  $C$  such that if  $K \in \mathcal{K}_0^n$ , there is an ellipsoid  $E \subset \mathbb{E}^n$  with  $\lambda_n(E) = \lambda_n(K)$  such that

$$\lambda_n(K \cap CE) \geq \lambda_n(K)/2.$$

In [621book, Proposition 5.5], it is proved that

$$\text{HYP} \Leftrightarrow \text{ELL}.$$

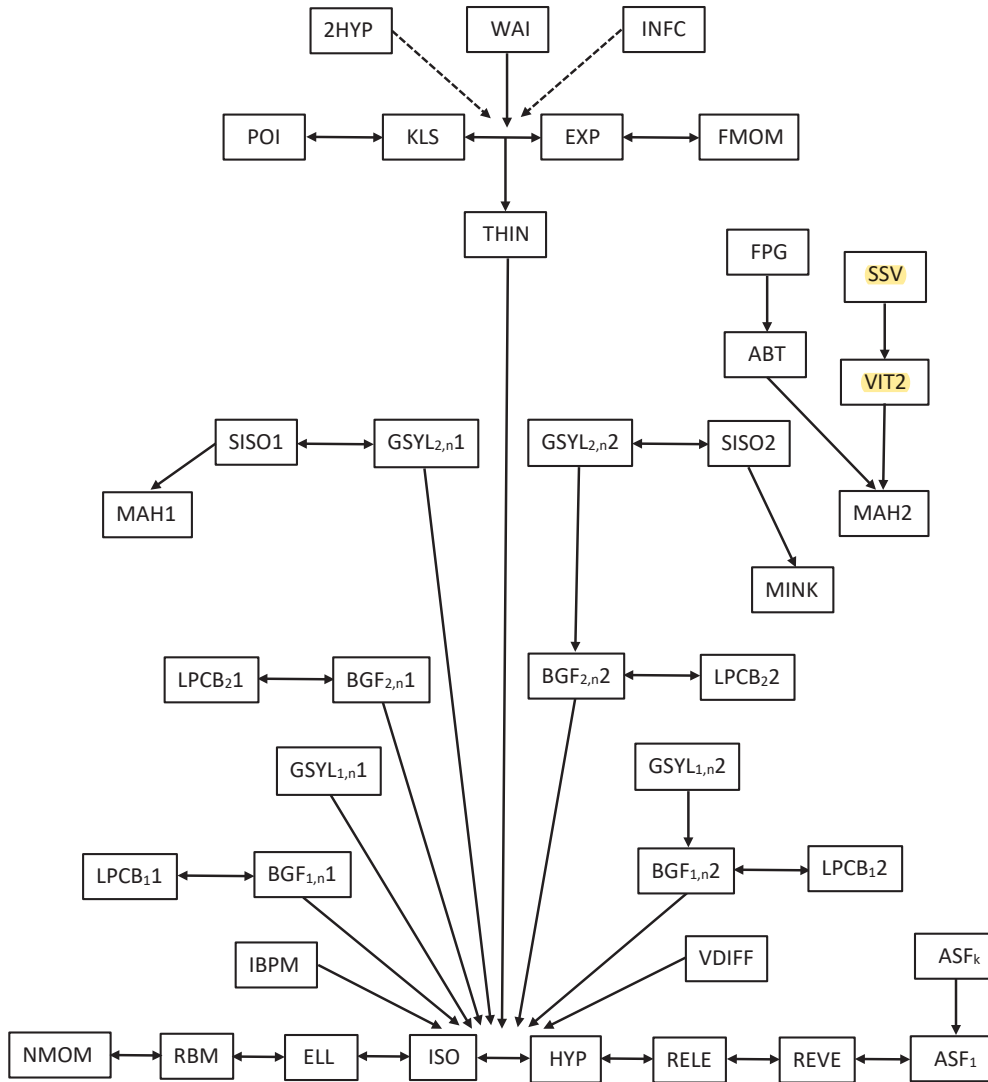


FIGURE 1. Relations between conjectures around the slicing problem or hyperplane conjecture (Problem 8.3) and Mahler’s conjecture (Problem 9.2). Dotted arrows indicate that only a symmetric version of  $KLS$  is known to follow.

Several more equivalent forms of HYP are known, set in the labyrinthine theory expounded in [16, Chapter 10] and [49], and surveyed in [139]. The following one was proposed by Dafnis and Paouris [64].

*Negative moment conjecture (NMOM)*. There are universal constants  $C, \xi > 0$  such that if  $K \in \mathcal{K}_0^n$  is isotropic, then

$$\max\{p \geq 1 : M_2(K) \leq \xi M_{-p}(K)\} \geq Cn,$$

where

$$M_p(K) = \left( \int_K \|x\|^p dx \right)^{1/p}.$$

Note that NMOM can be formulated in terms of dual volumes of  $K$ , since using (A.63), p. 412, we obtain

$$M_p(K)^p = \frac{n}{n+p} \tilde{V}_{n+p}(K).$$

By [64, Theorem 6.5] (see also [49, Theorems 6.4.2 and 6.4.3]), we have

$$\text{HYP} \Leftrightarrow \text{NMOM}.$$

Two further equivalent statements come from information theory. If  $X$  is a random vector in  $\mathbb{E}^n$  with probability density function  $f$ , its *differential entropy*  $h(X) = h(f)$  is defined by (5). The notation  $\text{Ent}(X)$  is also often used.

*Relative entropy conjecture (RELE)*. There is a universal constant  $C > 0$  such that if  $X$  is a random vector in  $\mathbb{E}^n$  whose probability density function  $f$  is log-concave, then

$$(6) \quad D(X) = D(f) = h(G) - h(X) \leq Cn,$$

where  $G$  is a Gaussian random vector with the same covariance matrix as  $X$ .

Bobkov and Madiman [41] prove that

$$\text{HYP} \Leftrightarrow \text{RELE},$$

giving credit on [43, p. 3320] to Ball for the idea of such an equivalence. See also [42, Conjectures V.4 and V.5] for this and a further conjecture equivalent to RELE, and [44] for related results. Madiman, Nayar, and Tkocz [162, Section 3] note that the extremal functions  $f$  for (6) differ from those for ISO.

The *entropy power* of a random variable  $X$  is defined by  $N(X) = (2\pi e)^{-1} e^{2h(X)/n}$ . The famous entropy power inequality of Shannon (see [264book, (55), p. 384]) states that if  $X$  and  $Y$  are independent random vectors in  $\mathbb{E}^n$  with probability densities in  $L^p(\mathbb{E}^n)$  for some  $p > 1$ , then  $N(X + Y) \geq N(X) + N(Y)$ .

*Reverse entropy power inequality conjecture (REVE)*. There is a universal constant  $C > 0$  such that if  $X_1, \dots, X_k$  are i.i.d. random vectors in  $\mathbb{E}^n$  whose probability density function is the density of an isotropic log-concave measure, then

$$N(X_1 + \dots + X_k) \leq C(N(X_1) + \dots + N(X_k)) = CkN(X_1).$$

Marsiglietti and Kostina [165, Theorem 1] (compare [41, Theorem 3]) prove that REVE  $\Leftrightarrow$  RELE and hence

$$\text{HYP} \Leftrightarrow \text{REVE}.$$

Further versions of HYP continue to appear due to the fundamental significance of the isotropic constant. For example, Fresen [83] finds a reformulation in terms of the convex floating body, and Alonso-Gutiérrez and Brazitikos [12, Theorem 1.1] provide an equivalent statement in terms of the optimal constant  $a(K)$ , where  $K \in \mathcal{K}_0^n$  is centered, for which there is an orthonormal basis  $\{u_1, \dots, u_n\}$  of  $\mathbb{E}^n$  such that the reverse dual Loomis–Whitney inequality

$$\lambda_n(K)^{n-1} \leq a(K) \prod_{j=1}^n \lambda_{n-1}(K \cap u_j^\perp)$$

holds (compare Meyer’s inequality in Note 9.7). Berndtsson, Mastrantonis, and Rubinstein [34] show that HYP would follow from their lower bounds for the  $L^p$ -Mahler volume (defined in the report below for Problem 9.2), if a certain convexity condition holds (the Ricci curvature of Bergman metrics of tube domains over a convex body has an upper bound independent of  $n$ ). We mention in passing articles by Brazitikos and Liakopoulos [50] and Liakopoulos [160], where versions of HYP for the surface area and other quermassintegrals of central sections of convex bodies are considered.

Bourgain’s  $L_K \leq Cn^{1/4} \log n$  bound for centered bodies  $K$ , mentioned in [88book, Note 8.9], is presented in [16, Theorem 10.3.3], [49, Section 3.3 and p. 135], and [89, Sections 2.4 and 2.5]. It was improved to  $L_K \leq Cn^{1/4}$  by Klartag [128], and this holds even for general bodies (again by [127, Remark 2, p. 392] or [190], see also [16, Theorem 10.5.4]). This was generalized and given an alternative proof by Klartag and Milman [138, Theorem 1.1] (see also [49, Theorems 7.3.2 and 7.5.15]). A spectacular advance due to Chen [60] in his work on the KLS conjecture (see below) led to the bound  $L_K \leq Cn^{o(1)}$ , asymptotically better than any positive power of  $n$ . Subsequent work follows Chen’s approach. A further significant improvement was made by Klartag and Lehec [136], who proved that  $L_K \leq C(\log n)^4$ . The exponent was slightly lowered by Jambulapati, Lee, and Vempala [117], but the best current bound of  $L_K \leq C\sqrt{\log n}$  is due again to Klartag [133]. This translates to a bound of  $\lambda_{n-1}(K \cap H) \geq C/\sqrt{\log n}$  for HYP.

The conjectures have been verified for various classes of convex bodies (meaning that for each class,  $L_K$  is bounded by a universal constant for each member  $K$  of the class), listed with references in [89, p. 65] and [145, p. 566], and often treated in [49], to which we shall refer when possible for details and further citations. As well as projection bodies (see also [49, Section 4.2.3]) and intersection bodies (and therefore polar projection bodies), already mentioned in Note 9.8, HYP is known to be true for unit balls of subspaces of  $L^p$ , for a fixed  $0 \leq p < \infty$  [23book], [617book]; unit balls of subspaces of quotients of  $L^p$ ,  $1 < p \leq \infty$  [411book], [617book]; unconditional convex bodies [49, Section 4.1] and centered convex bodies within a given Banach–Mazur distance of them [49, Corollary 4.2.6(ii)]; centered convex bodies with outer volume ratio bounded above by a fixed constant [49, Proposition 4.2.1]; unit balls of 2-convex spaces with fixed constant [49, Section 4.2.2]; unit balls of the Schatten classes



[49, Section 4.3]; convex polytopes with no more than a given number of vertices or facets [49, Section 4.4]; and  $k$ -intersection bodies [144]. Klartag and Kozma [135] (see also [49, Section 11.4]) verified the conjectures for certain random polytopes; more specifically, they proved that if  $K$  is the convex hull of  $m \geq n$  independent standard Gaussian vectors in  $\mathbb{E}^n$ , then  $L_K < C$  with probability at least  $1 - Ce^{-cn}$ , where  $c$  and  $C$  are universal constants (also not depending on  $m$ ). This was followed by a series of related works, to which references can be found in one of the most recent, that by Prochno, Thäle, and Turchi [194] (see also [49, Sections 11.5 and 11.6]). In some cases, the classes of bodies mentioned in this paragraph also satisfy certain of the potentially stronger conjectures in Figure 1 or represent positive partial answers to the measure slicing problem, as described below.

Is HYP true? Until recently, opinions differed among the leading experts, even (see [16, p. 360]) coauthors of the same book. The estimates following Chen's work in [60] are certainly encouraging. If HYP is false, all the other conjectures in Figure 1 also fail, but the many partial results and relations between the various conjectures would nevertheless fully justify the investigation.

Before turning to other conjectures stronger than HYP, we mention a discrete version of HYP and also explain how HYP bears on some other well-known open questions. The discrete version was posed by Koldobsky at an AIM meeting in 2013. It asks for the best constant  $c_n$  such that for every centered convex body in  $\mathbb{E}^n$  with  $\dim(K \cap \mathbb{Z}^n) = n$ , there is an  $S \in \mathcal{G}(n, n-1)$  such that

$$|K \cap \mathbb{Z}^n| \leq c_n |K \cap \mathbb{Z}^n \cap S| \lambda_n(K)^{1/n}.$$

Currently the best estimate, due to Freyer and Henk [85], is  $c_n \leq C_1 n^{10/3} (\log n)^{C_2}$ , where  $C_1$  and  $C_2$  are universal constants. See also the discussion in [93, Section 8].

HYP has consequences for *Hadwiger's conjecture* from combinatorial geometry posed in 1957, which states that at most  $2^n$  translates of the interior of a convex body  $K$  in  $\mathbb{E}^n$  are needed to cover  $K$  itself, the upper bound  $2^n$  only required if  $K$  is a parallelepiped. It also appertains to *Ehrhart's volume conjecture* from 1964, that if  $K$  is a convex body in  $\mathbb{E}^n$  with centroid at  $o$  and such that  $\mathbb{Z}^n \cap \text{int } K = \{o\}$ , then  $\lambda_n(K) \leq (n+1)^n/n!$ , with equality if  $K$  is the simplex  $K = (n+1)\text{conv}\{e_1, \dots, e_n\} - (1, 1, \dots, 1)$ . Both conjectures have been confirmed when  $n = 2$  but are open for  $n \geq 3$ . Until recently, the best known general bounds for the two conjectures were

$$\binom{2n}{n} (n \log n + n \log \log n + 5n) = O(4^n \sqrt{n} \log n) \quad \text{and} \quad 4^n,$$

respectively. Huang, Slomka, Tkocz, and Vritsiou [108] improved these bounds by utilizing thin-shell estimates for log concave measures from [97]. Campos, van Hintum, Morris, and Tiba [54] follow their approach but instead use Klartag and Lehec's bound for HYP to do even better, obtaining the bound  $4^n \exp(-Cn/(\log n)^8)$  for both conjectures. (Of course, a further improvement results from using Klartag's bound for HYP in [133].) In fact, these estimates have a third ingredient in common, which is to obtain a good lower bound for  $\lambda_n(K \cap (-K))/\lambda_n(K)$ ; the minimum is thought to be attained by a simplex, yielding a lower bound of the order of  $(2/e)^n$ , but this has not been proved.

*Sharp isotropic constant conjectures.* If  $K \in \mathcal{K}_0^n$ , then  $L_K \leq L_{\Delta_n}$  (SISO1), where  $\Delta_n$  is a regular simplex with centroid at the origin and unit volume. If  $K$  is also centered, then  $L_K \leq L_{C_n}$  (SISO2), where  $C_n$  is a centered unit cube.

SISO2 is explicitly stated in [596book, p. 312], but Ball's suggestion in [23, p. 85] (see also [20book, p. 83]) is apparently its first appearance in print.

Computations show that

$$(7) \quad L_{\Delta_n} = \frac{(n!)^{1/n}}{(n+1)^{(n+1)/(2n)}\sqrt{n+2}} \quad \text{and} \quad L_{C_n} = 1/\sqrt{12};$$

the former is shown in [171, p. 88], while the latter is easily checked. Since  $L_{\Delta_n}$  is uniformly bounded in  $n$ , we have

$$\text{SISO1} \Rightarrow \text{HYP} \quad \text{and} \quad \text{SISO2} \Rightarrow \text{HYP}.$$

Of course, one can also ask to identify the bodies that achieve equality in the conjectured inequalities. Affirmative answers to SISO1 and SISO2 with the expected equality conditions are available when  $n = 2$ , via those to related conjectures described below. Building on work of Campi, Colesanti, and Gronchi [53], Meyer and Reisner [174] prove that for  $n \geq 2$ , any local maximizer of  $L_K$ , for either SISO1 or SISO2, cannot be  $C_+^2$  at any point of  $\partial K$ , while Rademacher [196, (2), p. 309] shows that if a maximizer of  $L_K$  for SISO1 is a simplicial polytope, then it must be a simplex.

Fradelizi and Marín Sola [79] propose the following functional versions of the sharp isotropic constant conjectures. Let

$$f_0(x) = \exp(-\sum_{i=1}^n x_i) 1_{[-1, \infty)}(x), \quad f_1(x) = \exp(-\sum_{i=1}^n |x_i|), \quad \text{and} \quad f_\infty(x) = 1_{[-1, 1]^n}(x).$$

They conjecture that if  $f$  is a nonnegative log-concave function on  $\mathbb{E}^n$  with nonzero integral, then  $L_f \leq L_{f_0} = 1$  and  $\widehat{L}_f \leq \widehat{L}_{f_0} = e^{-1}$ . Denote these statements by  $P(n)$  and  $\widehat{P}(n)$ , respectively. They further conjecture that if  $f$  is also even, then  $L_f \leq L_{f_1} = 1/\sqrt{2}$  and  $\widehat{L}_f \leq \widehat{L}_{f_\infty} = 1/\sqrt{12}$ . Denote these statements by  $Q(n)$  and  $\widehat{Q}(n)$ , respectively. All four statements are true when  $n = 1$ ; see the references provided in [79]. It is proved in [79] that if  $\widehat{P}(n+1)$  holds, then SISO1 holds in  $\mathbb{E}^n$ , and conversely, if SISO1 holds in  $\mathbb{E}^{n+m}$  for all  $m \in \mathbb{N}$ , then  $\widehat{P}(n)$  holds. It is also shown that if  $\widehat{Q}(n)$  is true, then SISO2 holds in  $\mathbb{E}^n$ , and conversely, if SISO2 holds in  $\mathbb{E}^{n+m}$  for all  $m \in \mathbb{N}$ , then  $\widehat{Q}(n)$  is true.

In the context of convex bodies, all the above conjectures are inextricably linked with moments of volumes of random convex polytopes contained in a convex body. To explain this, for a bounded Borel set  $A$  in  $\mathbb{E}^n$ ,  $k \geq n$ , and  $p > 0$ , define

$$h_{p,k}(A) = \int_A \cdots \int_A \lambda_n(\text{conv}\{x_0, x_1, \dots, x_k\})^p dx_0 \cdots dx_k$$

and, when  $A$  is centered,

$$j_{p,k}(A) = \int_A \cdots \int_A \lambda_n(\text{conv}\{\pm x_1, \dots, \pm x_k\})^p dx_1 \cdots dx_k.$$

These quantities often appear in normalized form or modified as a  $p$ th mean, and with different notation. Their relevance stems from the formulas

$$(8) \quad L_K = \left( \frac{n! h_{2,n}(K)}{(n+1)\lambda_n(K)^{n+3}} \right)^{1/2n} \quad \text{and} \quad L_K = \frac{1}{2} \left( \frac{n! j_{2,n}(K)}{\lambda_n(K)^{n+2}} \right)^{1/2n},$$

for  $K \in \mathcal{K}_0^n$  and centered  $K \in \mathcal{K}_0^n$ , respectively. These are stated by Meckes [171, p. 88], [596book, p. 312], the latter providing references. For example, Ball [20book] utilizes  $j_{2,n}(K)$ . From (8) and Stirling's formula, we see that ISO, and hence HYP, is equivalent to  $h_{p,n}(K)^{1/np} \leq C/\sqrt{n}$  or  $j_{p,n}(K)^{1/np} \leq C/\sqrt{n}$  when  $\lambda_n(K) = 1$  and  $p = 2$ , but Meckes [171, p. 88] observes that this is true for any  $p \geq 1$ .

Slightly adapting the terminology of Meckes [596book], we shall refer to the problem of finding, for each  $n \geq 2$ ,  $k \geq n$ , and  $p \geq 1$ , those convex bodies (or centered convex bodies)  $K$  for which  $h_{p,k}(K)$  (or  $j_{p,k}(K)$ , respectively) achieve their extremal values, as the *generalized Sylvester problem*. The case  $k = n = 2$ ,  $p = 1$  arose from a discussion in the *Educational Times* of 1864–5 about the earlier version of Sylvester's problem stated in Note 9.4. Pfeifer [193] provides a historical account. It is known that the lower bounds are attained precisely by ellipsoids; the corresponding inequality  $h_{p,k}(K) \geq h_{p,k}(B^n)$ , where  $\lambda_n(K) = \kappa_n$ , is the *Blaschke–Groemer inequality* (see [216, Theorem 10.3.4]), and Meckes [171, Theorem 1(2)] provides the corresponding inequality for  $j_{p,k}(K)$ .

*Generalized Sylvester conjectures.* If  $K \in \mathcal{K}_0^n$  has unit volume,  $k \geq n$ , and  $p \geq 1$ , then  $h_{p,k}(K) \leq h_{p,k}(\Delta_n)$  (GSYL <sub>$p,k$</sub> 1) and if  $K$  is also centered, then  $j_{p,k}(K) \leq j_{p,k}(C_n)$  (GSYL <sub>$p,k$</sub> 2).

The special case GSYL<sub>1, $n$</sub> 1 has also been called the *simplex conjecture*. An interesting result of Bárány and Buchta [28] in the same direction shows that for each  $K \in \mathcal{K}_0^n$  with  $\lambda_n(K) = 1$ , there is an  $N = N(K) \geq n$  such that  $h_{1,k}(K) \leq h_{1,k}(\Delta_n)$  for all  $k \geq N$ .

Again, it can be asked to find the bodies that achieve equality in the conjectured inequalities. From (8), it is evident that

$$\text{GSYL}_{2,n}1 \Leftrightarrow \text{SISO1} \quad \text{and} \quad \text{GSYL}_{2,n}2 \Leftrightarrow \text{SISO2}.$$

An affirmative answer to GSYL <sub>$p,k$</sub> 1 in  $\mathbb{E}^2$  was provided by Campi, Colesanti, and Gronchi [53], generalizing earlier work of Blaschke (see [49, p. 135] and [216, pp. 540–1]) and Dalla and Larman [66]. Saroglou [211] proves that when  $k = n = 2$  and  $p \geq 1$ , simplices are the only maximizers, extending the result for  $p = 1$  of Giannopoulos [88]. Campi, Colesanti, and Gronchi [53] employ shadow systems, by means of which they also show that maximizers cannot be too smooth; specifically, their boundaries cannot be of class  $C_+^2$  in a nonempty relatively open subset (see also [49, Theorems 3.2 and 3.3]). Meckes [596book, Theorem 9] adapts the tools developed in [53] to obtain analogous results for centered bodies, in particular an affirmative answer to GSYL <sub>$p,k$</sub> 2 in  $\mathbb{E}^2$ .

For a bounded Borel set  $A$  in  $\mathbb{E}^n$ ,  $k \geq n$ , and  $p > 0$ , define

$$g_{p,k}(A) = \int_A \cdots \int_A \lambda_n(\text{conv}\{o, x_1, \dots, x_k\})^p dx_1 \cdots dx_k.$$

Compare the set function  $g_{m,k}$  defined on p. 353, which features in the Busemann random simplex inequality, Theorem 9.2.6; here we are abusing notation, but  $g_1 = g_{1,n}$  is the same in both cases. The lower bound for  $g_{p,k}$  on convex bodies is attained precisely for centered ellipsoids, by the *Busemann–Groemer inequality* [216, Theorem 10.3.5].

*Busemann–Groemer functional conjectures.* If  $K$  is a convex body of unit volume with centroid at the origin in  $\mathbb{E}^n$ ,  $k \geq n$ , and  $p \geq 1$ , then  $g_{p,k}(K) \leq g_{p,k}(\Delta_n)$  (**BGF** <sub>$p,k$</sub> 1). If  $K$  is also centered, then  $g_{p,k}(K) \leq g_{p,k}(C_n)$  (**BGF** <sub>$p,k$</sub> 2).

In [49, Proposition 3.5.2], it is shown that if  $p \geq 1$  and  $K \in \mathcal{K}_0^n$  is centered and of volume 1, then

$$(9) \quad g_{p,n}(K) \leq h_{p,n}(K) \leq (n+1)^p g_{p,n}(K).$$

The left-hand inequality in (9) yields the relation

$$\text{GSYL}_{p,n}2 \Rightarrow \text{BGF}_{p,n}2.$$

Moreover, [49, Corollary 3.5.8] uses (9) to conclude that if the simplex conjecture is true, the hyperplane conjecture follows, i.e.,

$$\text{GSYL}_{1,n}1 \Rightarrow \text{HYP}.$$

The previous implication and (9) are due to Giannopoulos and first appeared in his 1993 PhD thesis.

We shall continue to consider the Busemann–Groemer functional conjectures together with the following related ones, in which  $\Gamma_p K$  denotes the centroid body of  $K$ , defined in Note 9.5.

*$L_p$ -centroid body conjectures.* If  $K$  is a convex body of unit volume with centroid at the origin in  $\mathbb{E}^n$  and  $p \geq 1$ , then  $\lambda_n(\Gamma_p K) \leq \lambda_n(\Gamma_p \Delta_n)$  (**LPCB** <sub>$p$</sub> 1). If  $K$  is also centered, then  $\lambda_n(\Gamma_p K) \leq \lambda_n(\Gamma_p C_n)$  (**LPCB** <sub>$p$</sub> 2).

We adopt the usual custom of writing  $\Gamma K$  for  $\Gamma_1 K$ . A connection to the hyperplane conjecture is explicitly noted by Milman and Pajor [621book, Proposition 5.4], who show that **HYP** is equivalent to proving that  $\lambda_n(\Gamma K)^{1/n} \leq C/\sqrt{n}$  when  $K$  is centered and  $\lambda_n(K) = 1$  or to proving that  $\lambda_n(\Gamma_2 K) \leq C$  when  $K$  is centered and  $\lambda_n(K) = 1$ . For  $p = 1, 2$ , it is known that when  $\lambda_n(K) = 1$ , there are constants  $c_{p,n}$  depending only on  $n$  and  $p$  such that  $\lambda_n(\Gamma_p K) = c_{p,n} g_{p,n}(K)^{1/p}$ . (See [148book, p. 133]; for  $p = 1$ , this is due to Petty (see Theorem 9.1.5 and Note 9.1), while the case  $p = 2$  goes back to Blaschke; for  $p \neq 1, 2$ , such formulas are apparently not available.) This leads to yet further statements equivalent to **HYP**, involving  $g_{1,n}(K)$  and  $g_{2,n}(K)$ . From these facts, it follows that for  $p = 1$  or  $2$ ,

$$\text{BGF}_{p,n}1 \Leftrightarrow \text{LPCB}_p1 \Rightarrow \text{HYP} \quad \text{and} \quad \text{BGF}_{p,n}2 \Leftrightarrow \text{LPCB}_p2 \Rightarrow \text{HYP}.$$

It is expected that the upper bound in each set of conjectures is attained precisely for simplices or parallelotopes, respectively. With this equality condition, **LPCB** <sub>$p$</sub> 2 was proposed for  $p = 1, 2$  by Bisztriczky and Böröczky [67book, Conjectures 1.2 and 2.2], who proved both cases, and hence **BGF** <sub>$p,k$</sub> 2 for  $p = 1, 2$ , when  $n = 2$ . (This is already partially mentioned in Note 9.4.) They also proposed a version of **LPCB** <sub>$p$</sub> 1 for  $p = 1, 2$ , with the weaker restriction  $o \in K$ , and proved that the corresponding inequalities hold when  $n = 2$ , with equality if and

only if  $K$  is a triangle with  $o$  as a vertex. As was mentioned in Note 9.5, Campi and Gronchi [147book] extend these results by means of shadow systems; their results are more general, but they include proofs of  $\text{LPCB}_p1$  and  $\text{LPCB}_p2$ , for all  $p \geq 1$  when  $n = 2$ , and hence  $\text{BGF}_{p,n}1$  and  $\text{BGF}_{p,n}2$  for  $n = 2$  and  $p = 1, 2$ , with equality conditions.

Problem 8.3 as stated is now often called the *isomorphic Busemann–Petty problem* and its equivalence to HYP was explained in Note 9.8. Other variants of the Busemann–Petty problem are not relevant to HYP and will not be discussed here, but some, leading to generalizations of HYP, originate in the ideas of Zvavitch [872book] (see also [465book, Section 5.4]) to consider the Busemann–Petty problem for arbitrary measures and of Koldobsky [141] to obtain stability versions. For a brief description, we first note that the version of HYP in (9.10), p. 385 is equivalent to asking if there is a universal constant  $C$  such that for  $1 \leq k \leq n-1$  and centered  $K \in \mathcal{K}_0^n$ , we have

$$\lambda_n(K)^{(n-k)/n} \leq C^k \max_{S \in \mathcal{G}(n, n-k)} \lambda_{n-k}(K \cap S).$$

Indeed, the previous inequality results from iterating (9.10), p. 385, which is the case  $k = 1$ . In [143, Problem 1], Koldobsky asks if there is a universal constant  $C$  such that for any  $1 \leq k \leq n-1$ , any centered  $K \in \mathcal{K}_0^n$ , and any even measure  $\mu$  in  $\mathbb{E}^n$  with nonnegative continuous density function  $f$ , we have

$$(10) \quad \mu(K) \leq C^k \max_{S \in \mathcal{G}(n, n-k)} \mu^S(K \cap S) \lambda_n(K)^{k/n},$$

where  $\mu^S$  is the measure in  $S$  with density  $f$  with respect to  $\lambda_{n-k}$ . One might call this the *measure slicing problem*. By taking  $f \equiv 1$ , we see that an affirmative answer would also dispose of HYP. Koldobsky [142] shows that (10) holds with a factor  $n^{k/2}$  on the right-hand side. (Chasapis, Giannopoulos, and Liakopoulos [57] demonstrate that this remains true for arbitrary  $K \in \mathcal{K}_0^n$  with  $o \in \text{int } K$  when  $f$  is only assumed to be nonnegative and locally integrable.) Koldobsky [143] proves that for each  $\alpha \in (0, 1)$ , there is a  $C = C(\alpha)$  such that (10) is true whenever  $k \geq \alpha n$ . Extending a result of Milman [617book, Corollary 5.4], Koldobsky [143, Corollary 1] also obtains (10) with  $K$  replaced by a centered star body  $L$  and with a factor  $d_{\text{ovr}}(L, \mathcal{BP}_k^n)$  on the right-hand side. The latter quantity is an *outer volume ratio distance*, defined for a star body  $L$  and class  $\mathcal{E}$  of star bodies in  $\mathbb{E}^n$  by

$$d_{\text{ovr}}(L, \mathcal{E}) = \inf \left\{ \left( \frac{\lambda_n(M)}{\lambda_n(L)} \right)^{1/n} : L \subset M, M \in \mathcal{E} \right\},$$

and  $\mathcal{BP}_k^n$  is the class of  $(k, n-k)$ -intersection bodies in  $\mathbb{E}^n$ , defined in Note 8.7 on p. 341, where it was noted that  $\mathcal{BP}_1^n = \mathcal{I}_n$ . In a survey, Koldobsky [145, p. 567] lists many other positive results concerning the measure slicing problem. However, it turns out that (10) is generally false, since Klartag and Livshyts [137], improving on an earlier construction of Klartag and Koldobsky [134], find an example showing that up to a universal constant, a factor  $\sqrt{n}$  must be inserted on the right-hand side of (10) for the inequality to hold generally when  $k = 1$ . Further results on the measure slicing problem may be found in [39], [148], and [225].

Despite the falsity of (10), the following conjecture, proposed as a problem in [149, p. 262], is open, where we use notation from the previous paragraph.

*Isomorphic Busemann–Petty conjecture for measures (IBPM).* Let  $K$  and  $L$  be centered convex bodies in  $\mathbb{E}^n$  and let  $\mu$  be an even measure in  $\mathbb{E}^n$  with a nonnegative continuous density function  $f$ . There is a universal constant  $C$  such that if  $\mu^{u^\perp}(K \cap u^\perp) \leq \mu^{u^\perp}(L \cap u^\perp)$  for all  $u \in S^{n-1}$ , then  $\mu(K) \leq C\mu(L)$ .

Here  $\mu^{u^\perp}$  is the measure in  $u^\perp$  with density  $f$  with respect to  $\lambda_{n-1}$ . Taking  $f \equiv 1$ , and bearing in mind the equivalence of the slicing and isomorphic Busemann–Petty problems, we see that

$$\text{IBPM} \Rightarrow \text{HYP}.$$

Koldobsky and Zvavitch [149, Theorem 1] prove that the hypothesis of IBPM yields the conclusion  $\mu(K) \leq d_{BM}(K, \mathcal{I}_n) \mu(L)$ , where  $d_{BM}(K, \mathcal{I}_n)$  is the Banach–Mazur distance from  $K$  to the class  $\mathcal{I}_n$  of intersection bodies in  $\mathbb{E}^n$ , and from this, the fact that  $B^n \in \mathcal{I}_n$ , and Theorem 4.2.12 (John’s theorem), that the weaker conclusion  $\mu(K) \leq \sqrt{n} \mu(L)$  is valid. Unlike the measure slicing problem, is not known whether  $\sqrt{n}$  is optimal, nor whether  $d_{BM}(K, \mathcal{I}_n)$  can be replaced by  $d_{ovr}(K, \mathcal{I}_n)$ , which would give the desired conclusion for unconditional convex bodies. Of course, a version of IBPM can be formulated for lower-dimensional sections, and this is considered by Chasapis, Giannopoulos, and Liakopoulos [57] and Giannopoulos and Koldobsky [90].

Thus we have seen that the measure slicing problem and IBPM may be fundamentally different from each other, unlike their counterparts in the special case when  $\mu = \lambda_n$ . However, Koldobsky, Paouris, and Zvavitch [146] find a common approach by assuming that  $\mu_1^S(K \cap S) \leq \mu_2^S(L \cap S)$ , for all  $S \in \mathcal{G}(n, n-k)$ , where  $\mu_1$  and  $\mu_2$  are measures with possibly different density functions  $f_1$  and  $f_2$ , and seeking an upper bound for  $\mu_1(K)$  in terms of  $\mu_2(L)$ . In this way, they are able to generalize several results from [39], [134], [137], and [143]. One could of course state stronger forms of IBPM involving two measures and/or lower-dimensional sections; we shall not do so here, but instead refer to Giannopoulos, Koldobsky, and Zvavitch [92] for the most general results in this direction. Still further variations on the theme of Busemann–Petty-type comparison theorems and slicing questions for functions are investigated by Giannopoulos, Koldobsky, and Zvavitch [93], Haddad and Koldobsky [100], and Koldobsky, Roysdon, and Zvavitch [147].

If  $L$  is a star body in  $\mathbb{E}^n$  and  $S \in \mathcal{G}(n, n-k)$ , where  $0 \leq k \leq n-2$ , define the *average section functional*  $\text{as}(L \cap S)$  by

$$\text{as}(L \cap S) = \int_{S^{n-1} \cap S} \lambda_{n-k-1}(L \cap S \cap u^\perp) du.$$

Note that  $\text{as}(L) = \text{as}(L \cap \mathbb{E}^n)$  corresponds to the case when  $k = 0$ . Also, up to a constant, the average section functional is a dual volume, since

$$\text{as}(L \cap S) = (n-k) \kappa_{n-k-1} \tilde{V}_{n-k}(L \cap S),$$

as can be seen by using (A.58), p. 410 (with  $n$  replaced by  $n - k$  and  $n - k - 1$ ) and Kubota's formula, Theorem A.2.7, and taking  $L = B^n$  to evaluate the constant. The following conjectures were posed as a question by Brazitikos, Dann, Giannopoulos, and Koldobsky [48, Question 1.1].

*Average section functional conjectures (ASF<sub>k</sub>).* If  $1 \leq k \leq n - 2$ , there is a universal constant  $C$  such that if  $K \in \mathcal{K}_0^n$  is centered, then

$$(11) \quad \text{as}(K) \leq C^k \lambda_n(K)^{k/n} \max_{S \in \mathcal{G}(n, n-k)} \text{as}(K \cap S).$$

In [48, Proposition 4.5], it is shown that

$$\text{ASF}_1 \Leftrightarrow \text{HYP}.$$

From earlier work of Koldobsky cited in [48] it is known that (11) holds in the class of intersection bodies. In [48, Theorem 1.3], it is proved that (11) is true, even for centered star bodies, with an extra factor  $d_{\text{ovr}}(L, \mathcal{BP}_k^n)$  on the right-hand side, yielding the conjecture for other classes besides. The authors also prove in [48, Theorem 1.6] that (11) holds with an extra factor  $L_K^k$  on the right-hand side, as well as several other related results.

A different strengthening of HYP is considered by Giannopoulos and Koldobsky [91, Question 1.1].

*Volume difference conjecture (VDIFF).* There is a universal constant  $C$  such that if  $1 \leq k \leq n - 1$  and  $\gamma_{k,n} = \min \gamma > 0$  satisfies

$$\lambda_n(K)^{(n-k)/n} - \lambda_n(L)^{(n-k)/n} \leq \gamma^k \max_{S \in \mathcal{G}(n, n-k)} (\lambda_{n-k}(K \cap S) - \lambda_{n-k}(L \cap S)),$$

for all centered  $K, L \in \mathcal{K}_0^n$  with  $L \subset K$ , then  $\sup_{n,k} \gamma_{n,k} \leq C$ .

Taking  $k = 1$ ,  $L = rB^n$ , and letting  $r \rightarrow 0$  yields inequality (9.10) on p. 385. In other words,

$$\text{VDIFF} \Rightarrow \text{HYP}.$$

Along with several other related results, it is shown in [91, Corollary 1.3] that the previous inequality holds with an extra factor of  $(\sqrt{n/k} (\log(en/k))^{3/2})^k$  on the right-hand side. **Extensions to arbitrary measures are discussed in [93, Section 7].**

A remarkable result of Klartag [132] establishes a direct connection between Problems 8.3 and 9.2. In [132, Theorem 1.1], he proves that if a convex body  $K$  in  $\mathbb{E}^n$  contains the origin in its interior and is a local minimizer of the volume product  $v(K) = \lambda_n(K)\lambda_n(K^*)$ , then

$$L_K L_{K^*} v(K)^{1/n} \geq \frac{1}{n+2}.$$

Consequently, SISO1 and (7) imply that

$$\frac{1}{(n+2)^n} \leq L_K L_{K^*} v(K) \leq L_{\Delta_n}^{2n} v(K) = \frac{(n!)^2}{(n+1)^{n+1} (n+2)^n} v(K)$$

and hence  $v(K) \geq (n+1)^{n+1} / (n!)^2 = v(\Delta_n)$ . Thus

$$\text{SISO1} \Rightarrow \text{MAH1},$$

the Mahler conjecture for bodies containing the origin in their interiors (see the report for Problem 9.2). A shorter proof of this result has been offered by Balacheff, Solanes, and Tzanev [21], and Fradelizi and Marín Sola [79] establish a functional version of it; see the report for Problem 9.2 below.

We have seen that (10) is generally false, so despite Figure 1, it is not true that any reasonably plausible conjecture stronger than HYP is open at the present time. Klartag [132, Section 5] also offers a cautionary note by constructing unconditional convex bodies that form counterexamples to [153, Conjecture 5.1]. The latter proposes that the maximum of

$$\phi(K) = \frac{1}{v(K)} \int_K \int_{K^*} (x \cdot y)^2 dx dy,$$

over all centered convex bodies in  $\mathbb{E}^n$ , where  $v(K)$  is the Mahler volume, is attained when  $K$  is an ellipsoid. Alonso-Gutiérrez [8] (see also [49, p. 137], [153, p. 890], and [216, pp. 577–8]) showed that this would imply both HYP and the Blaschke–Santaló inequality for centered convex bodies, Theorem 9.2.11, and moreover verified it for  $l_p^n$  balls.

The following conjecture is ascribed to Minkowski although apparently not published by him. It has important consequences in number theory, such as [170, Conjecture 1.2], and is known to be true when  $n \leq 10$ ; see [124], [217], and the references given there. Note that the stronger Woods’ conjecture, also proved for  $n \leq 10$  in [124], is now known to be false when  $n \geq 24$ .

*Minkowski’s conjecture (MINK)*. Let  $o \in \Lambda$  be an  $n$ -dimensional lattice in  $\mathbb{E}^n$  whose Voronoi cell at  $o$  has unit volume. Then for each  $x \in \mathbb{E}^n$ , there is a  $v = (v_1, \dots, v_n) \in \Lambda + x$  such that  $v_1 v_2 \cdots v_n \leq 2^{-n}$ .

Building on earlier results and the known connection [199, p. 592] between the two conjectures, Magazinov [163] proves that

$$\text{SISO2} \Rightarrow \text{MINK}.$$

To describe still further conjectures related to HYP, some terminology is required. Let  $\mu$  be a finite Borel measure in  $\mathbb{E}^n$ . If  $A$  is a Borel set in  $\mathbb{E}^n$ , its (lower) *outer Minkowski content with respect to  $\mu$*  is defined by

$$\mu^+(A) = \liminf_{\varepsilon \rightarrow 0^+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon},$$

where  $A_\varepsilon = \{x : d(x, A) < \varepsilon\}$ . If  $\mu$  is an isotropic log-concave measure in  $\mathbb{E}^n$ ,

$$(12) \quad \text{Is}_\mu = \inf_A \frac{\mu^+(A)}{\min\{\mu(A), 1 - \mu(A)\}} = \inf_{\{A: \mu(A) \leq 1/2\}} \frac{\mu^+(A)}{\mu(A)} = 2 \inf_{\{A: \mu(A) = 1/2\}} \mu^+(A)$$

is the *Cheeger constant* of  $\mu$ , where  $A$  represents any Borel set in  $\mathbb{E}^n$ . The term derives from Cheeger’s lower bound [59]  $\lambda_1 \geq h^2/4$  for the least nonzero eigenvalue  $\lambda_1$  of the Laplace–Beltrami operator on a Riemannian manifold in terms of its Cheeger constant  $h$ , defined in a way corresponding to (12). The right-hand equality in (12) is a consequence of the concavity of the *isoperimetric profile*  $I(\mu)$  of  $\mu$ , the pointwise maximal function  $I : [0, 1] \rightarrow [0, \infty)$  such that  $\mu^+(A) \geq I(\mu(A))$  for all Borel sets  $A$  in  $\mathbb{E}^n$ , proved in suitably general form by



Milman (see [175, Theorem 1.8] and the remarks that follow). In the special case when  $\mu$  is the normalized restriction of volume  $\lambda_n$  to a convex body  $K$  in  $\mathbb{E}^n$ , Jerison [118, p. 731] states that the infimum is achieved by an open set that attains

$$(13) \quad \min \{ \mathcal{H}^{n-1}((\partial A) \cap \text{int } K) : A \subset K \text{ is open and } \lambda_n(A) = \lambda_n(K)/2. \}$$

(The part of  $\partial A$  contained in  $\partial K$  is not counted by  $\mu^+$  since  $\mu$  vanishes outside  $K$ .)

Several variants of these concepts are found in the swampy literature surrounding them. Jerison works with Borel subsets  $A$  of a domain  $\Omega$  in  $\mathbb{E}^n$  and calls a set  $E \subset \Omega$  *isoperimetric in  $\Omega$*  if it attains

$$(14) \quad \inf \{ P(A)/\lambda_n(A) : \lambda_n(A) = \alpha \lambda_n(\Omega) \},$$

where  $0 < \alpha < 1$  is fixed and  $P(A)$  is the perimeter of  $A$ . The case  $\alpha = 1/2$  is equivalent to (13). In [118, Conjecture 2.3], he conjectures that when  $\Omega$  is convex, the boundaries in  $\Omega$  of open isoperimetric sets in  $\Omega$  are Lipschitz graphs. Elsewhere, such as [51] and [159], sets attaining the infimum in (14) with the ratio  $\lambda_n(A)/\lambda_n(\Omega)$  unrestricted are called *Cheeger sets*, and the infimum itself the Cheeger constant of  $\Omega$ , a quite different usage. The endeavor of finding, computing, and classifying Cheeger sets overlaps with the study of isoperimetric problems restricted to certain regions or spaces, surveyed by Ros [202].

The following conjecture originates in the work of Kannan, Lovász, and Simonovits [121] and is expounded at length in [9] and [49, Chapter 14] (see also [10] and [157] for short surveys).

*Kannan–Lovász–Simonovits conjecture (KLS)*. There is a universal constant  $C > 0$  such that

$$(15) \quad \text{Is}_n = \min \{ \text{Is}_\mu : \mu \text{ is an isotropic log-concave measure in } \mathbb{E}^n \} \geq C.$$

(Recently, authors have often preferred to work with  $\psi_\mu = 1/\text{Is}_\mu$  and  $\psi_n = 1/\text{Is}_n$  instead.)

KLS was motivated by various algorithmic problems, for example, computing the volume of a convex body; see, for example, [9, Section 1.2] and [157]. (Efficient algorithms that avoid appeal to KLS have been proposed by Cousins and Vempala [61].) This application derives from the following alternate form of the conjecture (see [71, p. 532] or [151, (1.2), p. 3759]): There is a universal constant  $C > 0$  such that for  $0 < \alpha \leq 1/2$  and a Borel subset  $A$  of a convex body  $K$  in  $\mathbb{E}^n$  with centroid at  $o$  such that  $\lambda_n(A) = \alpha \lambda_n(K)$ , we have

$$(16) \quad \mathcal{H}^{n-1}((\partial A) \cap \text{int } K) \geq \alpha C \inf_{u \in S^{n-1}} \lambda_{n-1}(K \cap u^\perp).$$

Taking  $\alpha = 1/2$ , this says that, up to a universal constant, the most efficient way to halve the volume of a convex body is by cutting it with a hyperplane. The algorithmic applications stem from the use of this fact in repeated bisections. Unfortunately, the equivalence of the formulation in (16), and the fact that it suffices to prove KLS when  $\mu$  is the normalized restriction of volume  $\lambda_n$  to a convex body in  $\mathbb{E}^n$ , appear to be folklore shared by a few experts, since no proofs have appeared in print.

As we shall explain, KLS sits in the center of a heap of conjectures, all at least as strong as HYP. We now list a few of these known to be equivalent to KLS, sticking to our Euclidean setting even though extensions to Riemannian manifolds have been considered.

*Poincaré inequality conjecture (POI)*. There is a universal constant  $C > 0$  such that for every isotropic log-concave measure  $\mu$  in  $\mathbb{E}^n$  and every smooth function  $\varphi$  such that  $\int_{\mathbb{E}^n} \varphi d\mu = 0$ , one has

$$(17) \quad C \int_{\mathbb{E}^n} \varphi^2 d\mu \leq \int_{\mathbb{E}^n} \|\nabla\varphi\|^2 d\mu.$$

*Exponential concentration conjecture (EXP)*. There is a universal constant  $C > 0$  such that if  $\mu$  is an isotropic log-concave measure in  $\mathbb{E}^n$  and  $g$  is a 1-Lipschitz function on  $\mathbb{E}^n$ , then

$$(18) \quad \mu(\{x : |g(x) - \mathbb{E}_\mu g| \geq t\}) \leq e^{1-Ct},$$

where  $\mathbb{E}_\mu$  denotes expectation with respect to  $\mu$ .

*First moment concentration conjecture (FMOM)*. There is a universal constant  $C > 0$  such that if  $\mu$  is an isotropic log-concave measure in  $\mathbb{E}^n$  and  $g$  is a 1-Lipschitz function on  $\mathbb{E}^n$ , then

$$(19) \quad C \int |g(x) - \mathbb{E}_\mu g| d\mu \leq 1,$$

where  $\mathbb{E}_\mu$  denotes expectation with respect to  $\mu$ .

*Thin shell conjecture (THIN)*. There is a universal constant  $C > 0$  such that if  $\mu$  is an isotropic log-concave measure in  $\mathbb{E}^n$ , then

$$(20) \quad \sigma_n = \left( \int_{\mathbb{E}^n} (\|x\| - \sqrt{n})^2 d\mu(x) \right)^{1/2} \leq C.$$

THIN is sometimes called the *variance conjecture*, because under its hypotheses, (20) is equivalent to  $\text{Var}_\mu \|X\|^2 \leq Cn$ , where  $X$  is a random variable whose probability density function is the density of  $\mu$ ; see [49, Lemma 12.3.1] or [73, Lemma 1.4]. (Perhaps for this reason, the constant  $\sigma_n$  has been defined differently as  $\sup_\mu \sigma_\mu$ , where  $n\sigma_\mu^2 = \text{Var}_\mu \|X\|^2$ . In the context of THIN, the two definitions are equivalent in view of [49, Lemma 12.3.1] or [73, Lemma 1.4].) The question is explicitly stated by Bobkov and Koldobsky [40, p. 46] and arose from the pursuit of a central limit theorem for convex bodies, as discussed briefly in Note 9.9 and finally achieved by Klartag [129]. Concerning the latter, further information is given in [16, pp. 363–5] and [49, Chapter 12], and Fresen [84] presents a short proof of Klartag’s theorem, as well as a proof that it is implied by THIN.

It is known that

$$\text{KLS} \Leftrightarrow \text{POI} \Leftrightarrow \text{EXP} \Leftrightarrow \text{FMOM} \Rightarrow \text{THIN} \Rightarrow \text{HYP},$$

and moreover that these relations hold when  $\mu$  is fixed.

To the three conjectures equivalent to KLS should be added another, due to Jiang, Lee, and Vempala [119]. This is slightly too technical to state here, but is, roughly speaking, a conjectured central limit theorem to the effect that for independent random vectors  $X, Y$  with log-concave probability densities, the random variable  $X \cdot Y$  is close to a Gaussian.

That  $\text{KLS} \Rightarrow \text{HYP}$  had apparently been observed by Ball by 2006, the argument appearing much later in joint work with Nguyen [25, Section 5]. In [25, Theorem 5.1] (see also [49,

Theorem 16.1.1], which we follow here), it is shown that if  $X, Y$  are independent random variables whose probability density function is the density of an isotropic log-concave measure  $\mu$ , and

$$(21) \quad h\left(\frac{X+Y}{\sqrt{2}}\right) - h(X) \geq \kappa(h(Z) - h(X)),$$

where  $0 < \kappa < 1$ ,  $h$  denotes entropy defined by (5), and  $Z$  is the standard Gaussian random vector, then the isotropic constant  $L_\mu \leq e^{1+2/\kappa}$ . The desired implication now follows from  $\text{KLS} \Rightarrow \text{POI}$  and [25, Theorem 1.1], the latter proving that if an isotropic log-concave measure  $\mu$  with density  $f$  satisfies (17), then (21) holds with  $\kappa = C/8$ .

Denote by  $\lambda_1(\mu)$  the best  $C$  such that (17) holds; see [49, Theorem 14.1.5] and the discussion preceding it. (The notation  $C_P(\mu) = 1/\lambda_1(\mu)$  is sometimes used.) When  $\mu$  is the normalized restriction of volume  $\lambda_n$  to a convex body  $K$  in  $\mathbb{E}^n$ ,  $\lambda_1(\mu) = \lambda_1(K)$ , the first eigenvalue of the Laplacian on  $K$ . It is known that

$$(22) \quad \frac{1}{4}\text{Is}_\mu^2 \leq \lambda_1(\mu) \leq \pi \text{Is}_\mu^2.$$

These estimates are given by Ledoux [155, (5.8), p. 238] (see also [49, Theorems 14.1.6 and 14.1.7]), with  $36$  instead of  $\pi$  on the right-hand side. As noted by Klartag [133, p. 3], the improved constant in (22) was found by De Ponti and Mondino [67]. The inequality on the left, a form of Cheeger's inequality  $\lambda_1 \geq h^2/4$  from [59] (proved independently by Maz'ya [168]), yields  $\text{KLS} \Rightarrow \text{POI}$ . That on the right, which originates in work of Buser [52], shows that  $\text{POI} \Rightarrow \text{KLS}$ . The implication  $\text{POI} \Rightarrow \text{EXP}$  was proved by Gromov and Milman [96]. That  $\text{EXP} \Leftrightarrow \text{FMOM} \Rightarrow \text{KLS}$  is due to Milman [175], who also shows that the best constants  $C$  in (18) and (19) equal  $\text{Is}_\mu$ , up to a universal constant; see also [9, Sections 1.4 and 1.5] and [49, Section 14.2], as well as [176] for more general versions of  $\text{FMOM} \Rightarrow \text{KLS}$ . To show that  $\text{POI} \Rightarrow \text{THIN}$ , take  $\varphi(x) = \|x\|^2 - n$  in (17), using (2) to check that  $\int_{\mathbb{E}^n} \varphi d\mu = 0$ , and then apply the inequality at the beginning of the proofs of [49, Lemma 12.3.1] or [73, Lemma 1.4] to conclude that

$$(23) \quad \sigma_n \leq 2\lambda_1(\mu)^{-1/2}.$$

Finally,  $\text{THIN} \Rightarrow \text{HYP}$  was demonstrated by Eldan and Klartag [73] (see also [9, Section 3.1] and [49, Theorem 12.5.1]), who prove that

$$L_\mu \leq C\sigma_n$$

for any isotropic log-concave measure  $\mu$  in  $\mathbb{E}^n$ . These are the only known relationships between these conjectures. However, Eldan [71] (see also [49, Section 14.6]) proved that

$$(24) \quad \text{Is}_n \geq C \left( (\log n) \sum_{k=1}^n \frac{\sigma_k^2}{k} \right)^{-1/2},$$

a strong result towards reversing  $\text{KLS} \Rightarrow \text{THIN}$ .

We mentioned above that it suffices to prove  $\text{KLS}$  or any of its equivalent statements when  $\mu$  is the normalized restriction of volume  $\lambda_n$  to a convex body  $K$  in  $\mathbb{E}^n$ . Kolesnikov and Milman

[150, Theorem 12] show that in POI one can further assume that  $\varphi$  is a harmonic function on  $K$ . Unlike HYP, for which several reductions have been established (see, for example, [49, Section 6.2]), little else seems to be known regarding reductions of KLS.

Next, we summarize the evidence for these conjectures. Lee and Vempala [156], who also provide references for earlier bounds obtained by several authors (see also [49, Chapter 13] and [97, pp. 1044–5]), use a variant of the so-called stochastic localization scheme introduced by Eldan in proving (24). (For more about the latter method, see [72].) Their result is  $\sigma_n \leq Cn^{1/4}$ , which together with (24) gives  $Is_n \geq Cn^{-1/4}(\log n)^{-1/2}$ . However, in a remarkable development in which he builds on the methods of [71] and [156], Chen [60] proves that  $Is_n \geq Cn^{-o(1)}$ . In fact, Chen actually first shows that  $\sigma_n \leq Cn^{o(1)}$ , the lower bound for  $Is_n$  following from (24) at the cost of a factor of  $\log n$ . By adding other techniques to Chen’s method, substantial improvements on his bounds were made by Klartag and Lehec [136] (see also Jambulapati, Lee, and Vempala [117]). Like Chen, these latter authors obtained an upper bound for  $\sigma_n$  first. Klartag [133] again uses Chen’s method, but instead applies a refined version of an inequality due to Lichnerowicz (bounding  $\lambda_1(\mu)$  from below for so-called  $t$ -uniformly log concave measures) to estimate  $Is_n$  directly. His bound  $Is_n \geq C/\sqrt{\log n}$  yields  $\sigma_n \leq C\sqrt{\log n}$  (via (23) and the left inequality in (22)) and these are the best estimates currently available.

Kolesnikov and Milman [151, p. 3581] provide a list of references to works that establish KLS for various classes of convex bodies (i.e., for the normalized restriction of volume  $\lambda_n$  to members of these classes), to which they add generalized Orlicz balls (see also [30]). Due to the relations between the constants in KLS, POI, EXP, FMOM, THIN, and HYP described above, proving any of the first four conjectures true for a class of convex bodies implies the same for the other conjectures, and HYP must be true for any class for which THIN is true. Interestingly, it is not known whether KLS is true for unconditional convex bodies (see [49, Section 14.5]), but Klartag [130] (see also [9, Section 2.4] and [49, Proposition 12.4.1]) proves THIN for this class. Alonso-Gutiérrez and Bastero [11] verify THIN for hyperplane projections and Steiner symmetrizations of unit balls in  $l_p^n$ ,  $p \geq 1$ , extending earlier work described in [49, Section 12.3]. Several other positive partial results are listed in [11, p. 881]; more recently, Radke and Vritsiou [197], Vritsiou [224], and Dadoun, Fradelizi, Guédon, and Zitt [63] have confirmed THIN for the unit balls of classical spaces of matrices with the operator norm, including the Schatten classes.

Alonso-Gutiérrez, Prochno, and Thäle [13, Theorem A] show that a certain statement (too technical to reproduce here) about moderate or large deviations for isotropic log-concave random vectors implies that KLS is false.

The following statement was posed as a question by Lee and Vempala [157, p. 29].

(LSOB). There is a universal constant  $C > 0$  such that for every isotropic log-concave measure  $\mu$  in  $\mathbb{E}^n$  and every smooth function  $\varphi$  such that  $\int_{\mathbb{E}^n} \varphi^2 d\mu = 1$ , one has the log-Sobolev inequality

$$(25) \quad C \operatorname{Ent}_{\mu}(\varphi^2) \leq \int_{\mathbb{E}^n} \|\nabla \varphi\|^2 d\mu,$$

where

$$\text{Ent}_\mu(\varphi) = \int_{\mathbb{E}^n} \varphi \log \varphi \, d\mu - \left( \int_{\mathbb{E}^n} \varphi \, d\mu \right) \log \left( \int_{\mathbb{E}^n} \varphi \, d\mu \right).$$

Setting  $\varphi = 1 + \varepsilon g$  in (25) (cf. [203, p. 119]), it is not hard to check that

$$\text{Ent}_\mu(\varphi^2) = 2\varepsilon^2 \text{Var}_\mu(g) + O(\varepsilon^3),$$

where

$$\text{Var}_\mu(g) = \int_{\mathbb{E}^n} g^2 \, d\mu - \left( \int_{\mathbb{E}^n} g \, d\mu \right)^2,$$

and  $\int_{\mathbb{E}^n} \|\nabla \varphi\|^2 \, d\mu = \varepsilon^2 \int_{\mathbb{E}^n} \|\nabla g\|^2 \, d\mu$ . Letting  $\varepsilon \rightarrow 0$ , we see that the log-Sobolev inequality (25) implies Poincaré's inequality (17) (in a slightly more general form), so via POI, we obtain

$$\text{LSOB} \Rightarrow \text{KLS}.$$

Moreover the constant in (17) is the same as that in (25). Stavrakakis and Valettas [218] prove that LSOB holds for the class of  $l_p^n$  balls with  $2 \leq p \leq \infty$ . However, according to E. Milman (private communication), LSOB is actually false in general, failing for example for the exponential measure with density  $\|x\|_1/2^n$  and for normalized restriction of volume  $\lambda_n$  to  $B_1^n$ . See the article [158] by Lee and Vempala for related results.

There are at least three other conjectures that have been claimed to be at least as strong as KLS. To state the first, recall that if  $f$  and  $g$  are measurable functions on  $\mathbb{E}^n$ , their *infimal convolution* is defined by

$$f \square g(x) = \inf_{y \in \mathbb{E}^n} \{f(x-y) + g(y)\}.$$

The following conjecture from [154, p. 148] is sometimes called the Latała–Wojtaszczyk conjecture.

*Infimal convolution conjecture (INFC)*. There is a universal constant  $C > 0$  such that if  $\mu$  is an even log-concave measure in  $\mathbb{E}^n$  and  $\varphi : \mathbb{E}^n \rightarrow [0, \infty]$  is measurable, then

$$\left( \int_{\mathbb{E}^n} e^{\varphi \square (\Lambda^* \mu(\dot{\cdot}))} \, d\mu \right) \left( \int_{\mathbb{E}^n} e^{-\varphi} \, d\mu \right) \leq 1,$$

where

$$(26) \quad \Lambda^* \mu(x) = \sup_{y \in \mathbb{E}^n} \left\{ x \cdot y - \log \int_{\mathbb{E}^n} e^{y \cdot z} \, d\mu(z) \right\},$$

for  $x \in \mathbb{E}^n$ .

The function  $\Lambda^* \mu$  is sometimes called the *Cramer transform* of  $\mu$ . It is the Legendre transform (see [216, p. 40]) of the *logarithmic Laplace transform*  $\Lambda \mu$  of  $\mu$  (the second term on the right of (26)). The function  $\Lambda \mu$  is always convex with  $\Lambda \mu(o) = 0$ , and if  $\mu$  is log-concave, it is also  $C^\infty$ -smooth and strictly convex in the open set where it is finite; see [138, p. 16]. In particular,  $\Lambda^* \mu$  is a convex function. INFC is the focus of [49, Chapter 15], where it is noted that the conjecture is true for product measures, rotationally invariant measures, and uniform distributions on unit balls in  $l_p^n$ ,  $p \geq 1$ . It is claimed in [218, p. 363] that INFC implies KLS,

and this is repeated in [49, Theorem 15.3.13]. However, the proof of the latter requires that the measure  $\mu$  is even, so one can only conclude that INFC implies a symmetric version of KLS in which the measure  $\mu$  in (15) is even.

In fact, the basic question of whether KLS (or its equivalent version, see (16)) is equivalent to its symmetric version, in which the measure  $\mu$  in (15) is even (or the convex body  $K$  in (16) is centered, respectively), seems to be open at the present time. It is known that THIN is equivalent to its symmetric version; see [9, p. 46]. As observed by E. Milman, it follows from this and (24) that one can get from the symmetric version of KLS to KLS itself at the cost of a  $\log n$  factor.

If  $1 \leq m \leq n$ , the *lower Minkowski  $m$ -content* of a Borel set  $A$  in  $\mathbb{E}^n$  is defined by

$$M_*^m(A) = \liminf_{\varepsilon \rightarrow 0^+} \frac{\lambda_n(A + \varepsilon B^n)}{\kappa_{n-m} \varepsilon^{n-m}}.$$

See, for example, [75, p. 273].

*Waist conjecture (WAI)*. There exists a universal constant  $C > 0$  such that if  $1 \leq k \leq n$  and  $K \in \mathcal{K}_0^n$  with  $\lambda_n(K) = 1$ , there is a  $\phi \in SL_n$  such that if  $f : \phi K \rightarrow \mathbb{E}^k$  is continuous, there is an  $x \in \mathbb{E}^k$  for which

$$(27) \quad M_*^{n-k}(f^{-1}(x)) \geq C^k.$$

WAI is stated as a question by Klartag [131, p. 131]. He proves that (27) is true with  $C^k$  replaced by  $C^{m-k}$  on the right-hand side and states without proof that

$$\text{WAI} \Rightarrow \text{KLS}.$$

*Two hyperplane conjecture (2HYP)*. There exists a universal constant  $C > 0$  such that if  $K \in \mathcal{K}_0^n$  is centered and the open set  $E \subset K$  with  $\lambda_n(E) = \lambda_n(K)$  attains the infimum in (12), there is a half-space  $\Gamma$  such that

$$(\text{int } K) \cap \Gamma \subset E, \quad (\text{int } K) \cap (-\Gamma) \subset (\text{int } K) \setminus E, \quad \text{and} \quad \lambda_n(K \cap \Gamma) \geq C \lambda_n(K).$$

2HYP was proposed by Jerison [118, Conjecture 6.4]. Loosely speaking, it says that the interface between the isoperimetric set  $E$  for  $\text{int } K$  and the complement of  $E$  in  $\text{int } K$  is confined to the relatively small region between the hyperplanes that bound  $\Gamma$  and its reflection  $-\Gamma$  in the origin. From [118, Proposition 6.6], due to E. Milman, one can conclude that 2HYP implies the symmetric version of the alternate form (16) of KLS. The reader of [118, p. 729] should be aware that in [118], the latter is taken to be KLS itself, but, as was mentioned above, it is not known whether KLS is equivalent to its symmetric version.

### **Problem 8.5**

The problem remains open, but the following small observation may be relevant. Suppose  $L$  is a generalized intersection body. Then  $\rho_L = R\mu \in C(S^{n-1})$ , where  $\mu$  is a signed finite even Borel measure in  $S^{n-1}$ . Following the approach in the proof of Theorem 4.1.16, we can apply

the Jordan decomposition theorem to conclude that there are finite even Borel measures  $\bar{\mu}_j$  in  $S^{n-1}$ ,  $j = 1, 2$ , such that  $\mu = \bar{\mu}_1 - \bar{\mu}_2$ . If

$$(28) \quad R\bar{\mu}_j \in C(S^{n-1}),$$

and  $\rho_{L_j} = R\bar{\mu}_j$ , for  $j = 1, 2$ , then  $L_1$  and  $L_2$  are intersection bodies such that  $L_1 = L \tilde{+} L_2$ . Thus this problem has an affirmative answer whenever (28) holds.

### **Problem 8.6**

Until recently, there was one result, pertaining to the case when  $i = n - 1$ . (The statement in Note 8.6 concerning the case  $i = 1$  is erroneous.) Fish, Nazarov, Ryabogin, and Zvavitch [76] prove that if  $L$  is a star body in  $\mathbb{E}^n$  sufficiently close to  $B^n$  in the Banach–Mazur metric, then  $I^m L \rightarrow B^n$  as  $m \rightarrow \infty$  in the Banach–Mazur metric. For such  $L$ , if  $I^m L = cL$  for some  $c > 0$  and  $m \in \mathbb{N}$ , then  $L$  is a centered ellipsoid.

However, when  $i = n - 1$ , a slight modification of the problem has now been completely solved by Milman, Shabelman, and Yehudayoff [177]. If  $n \geq 3$  and the hypothesis that  $I^2 L$  is homothetic to  $L$  is replaced by  $I^2 L = cL$  for some  $c > 0$  (a stronger, but more natural assumption), then [177, Theorem 1.1] states that  $L$  must be a centered ellipsoid. Moreover, the same result holds, modulo sets of measure zero, if  $L$  is a Borel set star-shaped with respect to the origin; see [177, Remark 1.4]. The proofs depend on a novel use of continuous Steiner symmetrization, first employed by Rogers [201] in proving the result often called the Brascamp–Lieb–Luttinger inequality. The authors point out that if  $n = 2$ , then  $I^2 L = 4L$  holds for any centered star body.

### **Problem 8.7**

Regarding the case  $i = 1$ , the statement in Note 8.6 is misleading. Grinberg and Zhang [337book, Corollary 9.8] show that when  $n \geq 3$  (the condition is omitted), if  $I_1 L = cL$  for some  $c > 0$  (note that this implies that  $L$  is centered), then  $L$  is a centered ball. However, this nice result has no bearing on Problem 8.6.

With the slightly stronger assumption that  $IL = cL$  for some  $c > 0$ , Milman, Shabelman, and Yehudayoff [177, Corollary 1.2] provide a complete solution: If  $n \geq 3$ , then  $L$  must be a centered ball. As with their solution of Problem 8.6, the same result holds, modulo sets of measure zero, if  $L$  is a Borel set star-shaped with respect to the origin; see [177, Remark 1.4]. When  $n = 2$ , we have  $IL = 2L$  whenever  $L$  is invariant under a rotation by  $\pi/2$  about the origin. The authors show that their result for  $n \geq 3$  is a consequence of their solution of Problem 8.6. Indeed, if  $IL = cL$  for some  $c > 0$ , then  $I^2 L = I(cL) = c^{n-1} IL = c^n L$ , so  $L$  is a centered ellipsoid by [177, Theorem 1.1]. Then  $L = \phi B^n$  for some  $\phi \in GL_n$ , so by Theorem 8.1.6, we have

$$c\phi B^n = cL = IL = |\det \phi| \phi^{-t} I B^n = |\det \phi| \kappa_{n-1} \phi^{-t} B^n.$$

Therefore  $\phi^t \phi B^n = (\kappa_{n-1}/c) |\det \phi| B^n$ , so up to scaling,  $\phi$  is an orthogonal transformation and hence  $L$  is a centered ball.

**Problem 8.8**

Gardner, Ryabogin, Yaskin, and Zvavitch [87] answer part (i) of this problem by constructing coaxial convex bodies of revolution in  $\mathbb{E}^n$ ,  $n \geq 3$ , such that one body is centered and the other is not centrally symmetric, while both have the same inner section function. Shortly after, Nazarov, Ryabogin, and Zvavitch [187] went further by constructing an asymmetric convex body of revolution in  $\mathbb{E}^n$ ,  $n \geq 3$ , with a constant inner section function, thereby solving part (ii) of the problem. (For  $n = 4$ , this was achieved by the same authors earlier in [186].)

**Problem 8.9**

Nazarov, Ryabogin, and Zvavitch [186] provide a strong negative answer to part (i) of this problem when  $n \geq 4$  is even, by constructing non-congruent coaxial bodies of revolution  $K$  and  $L$  with  $\Pi K = \Pi L$ ,  $CK = CL$ , and  $IK = IL$ . Parts (ii) and (iii) appear to be open.

**Problem 9.1**

The problem is also stated by Schneider [216, (10.80), p. 570], who provides further commentary in [216, Notes 2 and 6, pp. 576–8].

Petty's conjectured projection inequality remains open and is one of the great challenges in convex geometry. In view of the Blaschke–Santaló inequality for centered convex bodies, Theorem 9.2.11, it is stronger than the Petty projection inequality, Theorem 9.2.9. The fact that the extremal bodies must be zonoids  $K$  such that  $\Pi^2 K$  is homothetic to  $K$ , already mentioned in Note 9.4, is detailed in [216, pp. 570–1]. This connection allows Saroglou and Zvavitch [215, Theorem 1.3] to prove that in a certain technical sense (see the report for Problem 4.4 above and the remarks in [215, p. 616]), ellipsoids are locally extremal bodies.

Saroglou [212, Theorem 3] proves that the usual Steiner symmetrization technique cannot be applied, at least in the obvious way, by showing that for  $n \geq 3$ , there exist  $K \in \mathcal{K}_0^n$  and  $u \in S^{n-1}$  such that  $\lambda_n(\Pi(S_u K)) > \lambda_n(\Pi K)$ . He also proves that Petty's conjectured projection inequality holds in  $\mathbb{E}^3$  when  $K$  is a cone or a centered double cone.

**Problem 9.2**

Mahler's conjecture, still an outstanding open problem in convex geometry, has received a great deal of attention in recent years. Here we shall only update and add some details to the report of Schneider [216, pp. 564–6], who summarizes developments prior to 2014.

Mahler's conjecture has two flavors, one (MAH1) for convex bodies containing the origin in their interiors (not just those with centroid at the origin, as stated in Problem 9.2) and the other (MAH2) for centered bodies. As Schneider points out, Mahler [164] actually only published MAH2, while MAH1 appeared in print much later, in [5, p. 150] and [98, (10.9), p. 59]. Some are of the opinion that MAH1 may be easier, because  $\Delta_n$ , a regular simplex in  $\mathbb{E}^n$  with unit volume and centroid at the origin, is thought likely to be the unique extremal body, up to a nonsingular linear transformation, while for MAH2, the extremal bodies are conjectured to be the Hanner polytopes (the term deriving from Hanner's article [103], but also sometimes associated to the names Hansen and Lima [104]), up to nonsingular linear transformations. We follow Schneider [216, pp. 564–5] in defining Hanner polytopes recursively: Closed line segments are Hanner polytopes, and a centered polytope in  $\mathbb{E}^n$  is a Hanner polytope if it is



either the convex hull of the union, or the sum, of two centered Hanner polytopes contained in complementary subspaces. In  $\mathbb{E}^2$  and  $\mathbb{E}^3$ , the only Hanner polytopes are centered cubes and octahedra, but there are many more in higher dimensions. To the early references provided in Note 9.4, it should be added that Meyer [173] also confirmed the anticipated equality case when  $n = 2$ .

The quantity  $v(K) = \lambda_n(K)\lambda_n(K^*)$  is often called the *volume product* or *Mahler volume* of  $K$ . The Bourgain–Milman bound mentioned in Note 9.4 is that  $v(K) \geq c_1^n v(C_n)$  when  $K$  is centered, where  $C_n = [-1/2, 1/2]^n$  and  $c_1$  is a universal constant, but Bourgain and Milman also showed that  $v(K) \geq c_2^n v(\Delta_n)$  for arbitrary  $K$  containing the origin in its interior, where  $c_2$  is a universal constant. Proofs are also given in [16, Theorems 8.2.2 and 8.5.1] and [49, Theorem 7.4.3]. See also the recent survey paper on the volume product by Fradelizi, Meyer, and Zvavitch [81], which overlaps significantly with the report below on the current status of Mahler’s conjecture; moreover, it covers upper as well as lower bounds for the volume product and gives some details of proofs of some of the main results.

Mahler’s conjecture received wide exposure from a 2007 post by Tao [219] in his blog. This article and the thread of comments it generated are still entertaining and informative. Even known results are continually being revisited. For example, both Meckes [172] and Saroglou [213] give different proofs of Reisner’s result for zonoids mentioned in Note 9.4, Meckes via Holmes–Thompson intrinsic volumes (see Note 4.1 for a special case) and Saroglou employing shadow systems. A probabilistic proof of Mahler’s conjecture for planar centered bodies is given by Tointon [221] and Rebello Bueno [198] finds a stochastic inequality in the plane that implies MAH2 when  $n = 2$ .

Proving MAH1 (or MAH2) up to a sub-exponential factor, i.e., showing that for all  $n \geq 2$ ,  $v(K) \geq c_n v(\Delta_n)$  (or  $v(K) \geq c_n v(C_n)$ , respectively) where  $c_n^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ , is enough to establish MAH1 (or MAH2, respectively) itself. This observation is due to Tao [220, p. 219], who uses the functional analogues just mentioned together with what he calls the “tensor power trick.” The same trick underlies a proof of this fact for MAH2 given independently by Berezovik and Karasev [31, Theorem 2.1].

In a remarkable work, Iriyeh and Shibata [110] prove MAH2 in  $\mathbb{E}^3$ , both the inequality and the fact that equality holds if and only if either  $K$  or  $K^*$  is a centered parallelepiped. Aspects of the Iriyeh–Shibata theorem are simplified by Fradelizi, Hubard, Meyer, Roldán-Pensado, and Zvavitch [78], who also provide a stability version. Apart from this, only the following partial results, which nevertheless represent a very significant effort, are known.

Kuperberg [153] proved the so-called bottleneck conjecture, which asserts that the volume of a certain domain  $K^\diamond \subset K \times K^* \subset \mathbb{E}^{2n}$ , where  $K \in \mathcal{K}_0^n$  is centered, is minimized if and only if  $K$  is an ellipsoid. (In its original form posed by Kuperberg [484book],  $K^\diamond = \text{conv}(K^+ \cup K^-)$ , where  $K^\pm = \{(x, y) \in K \times K^* : x \cdot y = \pm 1\}$ , while in [153], the convex hull is replaced by the union of line segments joining points in  $K^+$  to points in  $K^-$ .) This leads to improvements in the Bourgain–Milman bounds. In [153, Corollaries 1.6 and 1.8], Kuperberg shows that one can take  $c_1 = \pi/4$  in the centered case and  $c_2 = \pi/(2e)$  in the general case. These are currently the best bounds known when  $n \geq 4$ . Nazarov [184] found a completely different approach

based on multivariable complex and Fourier analysis, which yields the slightly weaker constant  $c_1 = (\pi/4)^3$  for the centered case. Variations of Kuperberg’s and Nazarov’s proofs, and commentary of various lengths, are provided by Berndtsson [32], [33], Błocki [36, Section 9], [37], [38], Fradelizi, Meyer, and Zvavitch [81], Mastrantonis and Rubinstein [167] (who also extend Nazarov’s method to the general case), and Ryabogin and Zvavitch [210]. (We note in passing that Bianchi and Kelly [35] find a proof of the Blaschke–Santaló inequality for centered convex bodies, Theorem 9.2.11, based on Nazarov’s approach.) Yet another way of recovering the Bourgain–Milman bounds, using techniques generally more familiar in convex geometry, is due to Giannopoulos, Paouris, and Vritsiou [94].

Nazarov, Petrov, Ryabogin, and Zvavitch [185] prove that  $C_n$  is a strict local minimizer for the volume product in the class of centered convex bodies endowed with the Banach–Mazur metric. Kim and Reisner [126] show that  $\Delta_n$  is a strict local minimizer in the class of all convex bodies, and Kim [125] significantly extends the result in [185] by showing that it holds not just for  $C_n$  but for all Hanner polytopes. Further evidence for the conjectured minimizers is obtained by Reisner, Schütt, and Werner [200], who prove that such a body cannot be  $C_+^2$  at any point of its boundary. Related results were obtained by Harrell, Henrot, and Lamboley [105]. Also related are the quantitative lower bounds for the volume product in terms of curvature obtained by Nakamura and Tsuji [183] via a connection to Ornstein–Uhlenbeck flow.

Mahler’s conjecture has been verified for some special classes of bodies. Summaries are given by Alexander, Fradelizi, García-Lirola, and Zvavitch [6], who study a related problem in finite metric spaces, and by Karasev [122], who employs symplectic methods to deal with the case of central hyperplane sections or projections of  $l_p$  balls or of Hanner polytopes. In addition to these and zonoids (see Note 9.4), the list includes bodies of revolution, polytopes (or centered polytopes) in  $\mathbb{E}^n$  with at most  $n + 3$  vertices (or at most  $2n + 2$  vertices, respectively), and unconditional bodies and some generalizations of them; see [20] and [81, Section 3.6] for references. Barthe and Fradelizi [29] confirm the conjecture for bodies invariant under the action of a Coxeter group. This line of investigation was continued by Iriyeh and Shibata [111], who study the variant of Mahler’s conjecture in which the minimum of  $v(K)$  is sought over those  $K \in \mathcal{K}_0^n$  that are invariant under a discrete subgroup  $G$  of  $O(n)$ . Note that MAH1 and MAH2 correspond to  $G = \{\text{Id}\}$  and  $G = \{\text{Id}, -\text{Id}\}$ , respectively, where  $\text{Id}$  is the  $n \times n$  identity matrix. They focus on the case  $n = 3$  and present the state of play in a table in [111, p. 5]. In [112], the same authors deal with the case when  $G$  is the special orthogonal group of the cube or simplex in  $\mathbb{E}^n$ . They obtain further results in the same vein in [113].

Functional versions of the inequalities at hand go back to Ball’s PhD thesis, where in [23, Corollary 4.5] he proved a functional version of Theorem 9.2.11, the Blaschke–Santaló inequality for centered convex bodies. Corresponding versions of the Bourgain–Milman theorem were established by Artstein-Avidan, Klartag, and Milman [18] and Fradelizi and Meyer [80], and in the latter paper the following functional analogues of MAH1 and MAH2 are proposed.

If  $\varphi : \mathbb{E}^n \rightarrow \mathbb{R} \cup \{\infty\}$  is a convex function such that  $0 < \int_{\mathbb{E}^n} e^{-\varphi(x)} dx < \infty$ , then

$$\int_{\mathbb{E}^n} e^{-\varphi(x)} dx \int_{\mathbb{E}^n} e^{-\mathcal{L}\varphi(x)} dx \geq e^n,$$

and for even  $\varphi$ ,  $e^n$  can be replaced by  $4^n$ .

Here  $\mathcal{L}\varphi$  is the Legendre transform of  $\varphi$ , defined for  $x \in \mathbb{E}^n$  by  $\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{E}^n} (x \cdot y - \varphi(y))$ . These conjectures are shown to be equivalent to MAH1 and MAH2 in [80, Section 2], and precise equality conditions are proposed that correspond to those conjectured for MAH1 and MAH2. Fradelizi and Nahkle [82] prove the conjectured inequality and equality condition for even functions when  $n = 2$ . The survey [81, Sections 4 and 5] (see also [216, pp. 523–4]) discusses various aspects of the functional inequalities in some detail, and corresponding inequalities involving more than one body or function are treated in [81, Section 6]. The Mahler conjecture for various classes of functions and in other settings are examined by Fradelizi, Gozlan, Sadovskiy, and Zugmeyer [77].

In the remainder of this report, we shall discuss various other results and conjectures related to MAH1 or MAH2.

Chan and Pak [56] note that an inequality for finite posets called the reverse Sidorenko inequality is a special case of MAH1. See also the article by Artstein-Avidan, Sadovskiy, and Sanyal [20].

Some  $L^p$  versions are formulated by Berndtsson, Mastrantonis, and Rubinstein [34], who introduce ideas of interest beyond Mahler's conjecture. If  $K$  is a compact (not necessarily convex) body in  $\mathbb{E}^n$  and  $p > 0$ , they define its  $L^p$ -support function  $h_{p,K}$  by

$$h_{p,K}(x) = \log \left( \frac{1}{\lambda_n(K)} \int_K e^{p \cdot x \cdot y} dy \right)^{1/p}$$

for  $x \in \mathbb{E}^n$ . In [34, Section 2], it is shown that  $h_{p,K}$  is a convex function (even if  $K$  is not convex), that  $h_{p,K} \leq h_{q,K} \leq h_K$  for  $p \leq q$ , and that  $\lim_{p \rightarrow \infty} h_{p,K} = h_K$ . In [34, Section 3], the  $L^p$ -polar body  $K^{*,p}$  of  $K$  is defined via the gauge function

$$\|x\|_{K^{*,p}} = \left( \frac{1}{(n-1)!} \int_0^\infty e^{-h_{p,K}(rx)} r^{n-1} dr \right)^{-1/n}$$

for  $x \in \mathbb{E}^n$ , so that  $K^{*,p} = \{x \in \mathbb{E}^n : \|x\|_{K^{*,p}} \leq 1\}$ . Using the fact that  $\int_0^\infty e^{-as} s^{n-1} ds = (n-1)! a^{-n}$ , one sees that  $\lim_{p \rightarrow \infty} \|x\|_{K^{*,p}} = h_K(x) = \|x\|_{K^*}$  and hence  $\lim_{p \rightarrow \infty} K^{*,p} = K^*$ . One also has  $K^* \subset K^{*,q} \subset K^{*,p}$  when  $p \leq q$ . Unlike the usual polar operation,  $^{*,p}$  is not a duality, since  $(K^{*,p})^{*,p} \neq K$ , in general, even when  $K$  is a centered convex body. With these concepts in hand, the authors define the  $L^p$ -Mahler volume  $\mathcal{M}_p(K)$  of  $K$  by

$$\mathcal{M}_p(K) = n! \lambda_n(K) \lambda_n(K^{*,p}) = \lambda_n(K) \int_{\mathbb{E}^n} e^{h_{p,K}(x)} dx.$$

Note that as  $p \rightarrow \infty$ ,  $\mathcal{M}_p(K)$  approaches  $n! v(K)$  from above; the authors justify the  $n!$  factor in their definition by noting that  $\mathcal{M}_p(K \times L) = \mathcal{M}_p(K) \mathcal{M}_p(L)$ . By [34, Lemma 4.6],  $\mathcal{M}_p(\phi K) = \mathcal{M}_p(K)$  for any  $\phi \in GL_n$ . Theorem 9.2.11, the Blaschke-Santaló inequality for

centered convex bodies, is generalized in [34, Theorem 1.6], which states that  $\mathcal{M}_p(K) \leq \mathcal{M}_p(B^n)$  for  $0 < p \leq \infty$ , but for finite  $p$  the precise equality condition remains open. In [34, Conjectures 1.3 and 1.4], the authors conjecture that when  $0 < p \leq \infty$ ,

$$(29) \quad \mathcal{M}_p(K) \geq \mathcal{M}_p(C_n)$$

for centered convex bodies  $K$  in  $\mathbb{E}^n$ , and

$$(30) \quad \mathcal{M}_p(K) \geq \mathcal{M}_p(\Delta_n)$$

for arbitrary convex bodies  $K$  in  $\mathbb{E}^n$ . (The case  $p = 1$  of (29) was first conjectured by Błocki [37, p. 56] and (30) appeared in a different form in [167, Conjecture 10].) They further conjecture that the lower bounds in (29) and (30) are attained if and only if  $K$  is an image of  $C_n$  (or  $\Delta_n$ , respectively) under a nonsingular linear transformation. In [34, Lemma 3.12], it is shown that (30) for all  $p \in (0, \infty)$  implies MAH1 and (29) for all  $p \in (0, \infty)$  implies MAH2.

The Mahler volume turns up in the following conjecture that arose from discussions between R. Freij, M. Henze, G. Kalai, M. Schmidt, and G. Zieger; see [74, Conjecture 10.1].

*If  $P$  is a centrally symmetric  $n$ -dimensional convex polytope in  $\mathbb{E}^n$ , then*

$$(31) \quad |\text{Flags}(P)| \geq \frac{(n!)^2}{2^n} v(P).$$

Here  $\text{Flags}(P)$  denotes the set of flags of  $P$ , i.e., sequences  $(F_0, F_1, \dots, F_n)$  of faces of  $P$  such that  $\dim F_i = i$  and  $F_i \subset F_{i+1}$  for  $i = 0, 1, \dots, n-1$ . Since MAH2 says that  $v(P) \geq 4^n/n!$ , it and (31) would prove Kalai's *flag number conjecture*, which reads as follows.

*If  $P$  is a centrally symmetric  $n$ -dimensional convex polytope in  $\mathbb{E}^n$ , then*

$$(32) \quad |\text{Flags}(P)| \geq 2^n n!.$$

This is a special case of [120, Conjecture C]; the general conjecture, as mentioned in [74, Introduction], is known to be false. Equality holds in (31) and (32) if and only if  $P$  is a Hanner polytope. The relationship between these conjectures is strengthened by a further one due to Faifman, Vernicos, and Walsh [74, Conjecture 1.1], set in the framework of Funk geometry:

*Let  $K$  be a convex body in  $\mathbb{E}^n$ , let  $r > 0$ , and let  $\text{Vol}_K(B_K(r))$  denote the (Holmes–Thompson) volume of the (forward) ball in the Funk geometry of radius  $r$  centered at  $o$ . Then  $\text{Vol}_K(B_K(r))$  is minimized when  $K$  is an  $n$ -dimensional simplex with centroid at  $o$ . If  $K$  is centrally symmetric,  $\text{Vol}_K(B_K(r))$  is minimized when  $K$  is a Hanner polytope.*

A Funk metric defined on the interior of a convex body in  $\mathbb{E}^n$  (not a true metric, but a non-reversible, i.e., not symmetric, pseudometric) gives rise to a particular type of Finsler geometry. The lack of symmetry necessitates defining both forward and backward balls. More detailed information and background may be found in [74], [81, Section 7.3], and [192], for example. It is known that as  $r \rightarrow 0$ ,  $\text{Vol}_K(B_K(r))$  is asymptotic to  $r^n v(K)$ , and in [74,

Theorem 1.2], it is proved that if  $P$  is an  $n$ -dimensional convex polytope, then

$$\lim_{r \rightarrow \infty} \frac{\kappa_n}{r^n} \text{Vol}_P(B_P(r)) = \frac{|\text{Flags}(P)|^2}{n!}.$$

It follows that [74, Conjecture 1.1] would yield MAH1 and MAH2 as  $r \rightarrow 0$  and (32) as  $r \rightarrow \infty$ . In [74, Theorem 1.3], the conjecture is proved true for unconditional convex polytopes.

The report for Problem 8.3, the slicing problem, includes the direct link found by Klartag [132], namely

$$\text{SISO1} \Rightarrow \text{MAH1},$$

where SISO1 denotes the sharp isotropic constant conjecture for arbitrary convex bodies. See Figure 1, which also illustrates the relations between other conjectures related to Mahler's conjecture. Fradelizi and Marín Sola [79] find a functional version of Klartag's result: If  $\widehat{P}(n)$  holds, then the functional version of MAH1 stated above also holds in  $\mathbb{E}^n$ . Here  $\widehat{P}(n)$  and a variation,  $P(n)$ , are the statements introduced shortly after that of SISO1 in the report for Problem 8.3. It is explained in [79] why this result does not work when  $\widehat{P}(n)$  is replaced by  $P(n)$ .

Iriyeh and Shibata [110, Section 1.2] note that their result on MAH2 bears on Viterbo's conjectured isoperimetric inequality for the symplectic capacities of convex bodies in  $\mathbb{E}^{2n}$  endowed with the standard symplectic structure. A *symplectic capacity*  $c$  is a set function with the following properties: monotonicity, invariance under symplectomorphisms, positive homogeneity of degree 2, and the normalization  $c(B^{2n}) = c(B^2 \times \mathbb{E}^{2n-2}) = \pi$ . The smallest symplectic capacity is the *Gromov width*

$$c_G(K) = \sup\{\pi r^2 : rB^{2n} \text{ embeds symplectically into } K\},$$

and the largest symplectic capacity is the *cylindrical capacity*

$$c_Z(K) = \inf\{R : K \text{ embeds symplectically into } Z^{2n}(R)\},$$

where  $Z^{2n}(R)$  is the cylinder  $\{z \in \mathbb{C}^n : \pi|z_1|^2 \leq R\}$ . Further examples are listed in [1] and [99], where more information and references can be found. Several non-equivalent symplectic capacities agree on convex bodies. In particular, [99, Theorem 1.12] (ascribed there to several different authors) implies that if  $K$  is a convex body in  $\mathbb{E}^{2n}$ , then

$$(33) \quad c_{HZ}(K) = c_{SH}(K) = c_{EH}^1(K),$$

where  $c_{HZ}$ ,  $c_{SH}$ , and  $c_{EH}^1$  are the *Hofer–Zehnder capacity*, the *Viterbo capacity*, and the first *Ekeland–Hofer capacity*, respectively. Their common value is often called the *Ekeland–Hofer–Zehnder capacity* and denoted by  $c_{EHZ}(K)$ . Formulas for this quantity, including a combinatorial expression when  $K$  is a polytope, are provided by Haim-Kislev [101].

Viterbo's conjecture [223, p. 426] is as follows.

*Viterbo's conjecture* (VIT). If  $K \in \mathcal{K}^{2n}$  and  $c$  is any symplectic capacity, then

$$(34) \quad \frac{c(K)}{c(B^{2n})} \leq \left( \frac{\lambda_{2n}(K)}{\lambda_{2n}(B^{2n})} \right)^{1/n}.$$

We shall label the *symmetric case of Viterbo's conjecture*, i.e., the conjecture restricted to centrally symmetric  $K \in \mathcal{K}^{2n}$ , by **VIT2**.

The surprising connection with Mahler's conjecture was discovered by Artstein-Avidan, Karasev, and Ostrover [17, Theorem 1.6], who showed that

$$\mathbf{VIT2} \Rightarrow \mathbf{MAH2}.$$

The proof follows easily from  $c(B^{2n}) = \pi$  and the fact, proved in [17, Theorem 1.7], that the **Hofer–Zehnder** capacity  $c_{HZ}$  of  $K \times K^*$ , where  $K \in \mathcal{K}^n$  is centered, equals 4. Conversely, **MAH2** implies **VIT2** when  $c = c_{HZ}$  for bodies of this form, so the Iriyeh–Shibata theorem yields this special case of **VIT2** when  $n = 3$ ; see [110, Corollary 1.3].

It was mentioned above that **MAH1** and **MAH2** are unaffected by the insertion of a sub-exponential factor on their right-hand sides. Berezovik and Karasev [31, Theorem 7.1] prove an analogous result for the following consequence of **VIT**:

$$(35) \quad \lambda_{2n}(K) \geq \frac{c_{EHZ}(K)^n}{n!},$$

where  $K$  is a convex body in  $\mathbb{E}^{2n}$  and  $c_{EHZ}$  is the Ekeland-Hofer-Zehnder capacity. The authors define the *symplectic polar body*  $K^\omega$  of a convex body  $K$  in  $\mathbb{E}^{2n}$  containing the origin in its interior by

$$K^\omega = \{x \in \mathbb{E}^{2n} : \omega(x, y) \leq 1 \text{ for all } y \in K\},$$

where  $\omega$  is the standard symplectic form on  $\mathbb{E}^{2n}$ . They note that  $K^\omega = JK^*$ , where  $J$  is the standard complex structure rotation, and conjecture that when  $K^\omega = K$  (a condition they show implies that  $K$  is centrally symmetric),  $\lambda_{2n}(K) \geq 2^n/n!$ . This conjecture is shown in [31, Theorem 4.3] to be equivalent to **MAH2**. In view of (35), the implication **VIT2**  $\Rightarrow$  **MAH2** follows if  $c_{EHZ}(K) \geq 2$  whenever  $K^\omega = K$ , but [31, Theorem 5.1] provides an even stronger result, that in fact  $c_{EHZ}(K) \geq 2 + 1/n$  whenever  $K$  is centrally symmetric and  $K^\omega \subset K$ .

**Viterbo** [223, Theorem 5.1] proved that (34) holds with a factor linear in  $n$  on the right-hand side. This was substantially improved by Artstein-Avidan, Milman, and Ostrover [19], who proved that (34) holds up to a universal constant factor. As noted in [2, p. 692], (34) holds when  $c = c_G$ , the Gromov width. It is also true for ellipsoids, and Abbondandolo and Benedetti [1, Corollary 2] prove that (34) holds for sufficiently smooth  $K$  in a neighborhood of  $B^{2n}$ . Cases where equality holds in (34) are discussed by Balitskiy [22] and Karasev and Sharipova [123]. Rudolf [205] finds reformulations of **VIT** that are variants of Moser's still-unsolved worm problem.

In [3, Section 4], Akopyan, Balitskiy, Karasev, and Sharipova show that **MAH1** does not follow from **VIT**. However, Haim-Kislev and Ostrover [102] prove that **VIT** is actually false for all  $n \geq 2$ . Before describing their counterexample, we mention another related conjecture.

The fact (see (33)) that several non-equivalent symplectic capacities agree on convex bodies suggests that perhaps they all do, and this conjecture has been around for some time (see [106, Conjecture 1.9], [169, Problem 53, p. 572], and [189, Conjecture 5.1]). According to Hermann [106, Remark 1.8], it is probably over 30 years old. Gutt, Hutchings, and Ramos [99] call it the *strong Viterbo conjecture*. They prove, among other results, that it holds in  $\mathbb{E}^4$  for a class



of bodies they call monotone toric domains. Cristoforo–Gardiner and Hind [62] generalize this result to higher dimensions. Edtmair [70] proves the conjecture for  $C^3$  convex bodies in  $\mathbb{E}^4$  in a neighborhood of  $B^4$ . However, the conjecture is false for all  $n \geq 2$ , because if it were true, the fact that Gromov width  $c_G$  satisfies (34) would imply that VIT is true. Nevertheless, the following remains a possibility.

*Symmetric strong Viterbo conjecture (SSV).* All symplectic capacities agree on centrally symmetric convex bodies.

Again since  $c_G$  satisfies (34), we have

$$\text{SSV} \Rightarrow \text{VIT2}.$$

Let  $P$  be the regular pentagon  $P = \text{conv} \{e^{2k\pi i/5}, k = 0, \dots, 4\} \subset \mathbb{E}^2$  and let  $\phi$  be the clockwise rotation by  $\pi/2$  about the origin in  $\mathbb{E}^2$ . Haim-Kislev and Ostrover [102, Proposition 1.3] find that

$$c_{EHZ}(P \times \phi P) = 2 \cos \frac{\pi}{10} \left(1 + \cos \frac{\pi}{5}\right).$$

Since  $\lambda_2(P) = (5/2) \sin(2\pi/5)$ , a computation yields

$$\frac{c_{EHZ}(P \times \phi P)}{2\lambda_4(P \times \phi P)} = \frac{c_{EHZ}(P \times \phi P)}{2\lambda_2(P)^2} = \frac{\sqrt{5} + 3}{5} > 1,$$

contradicting (34).

It is worth mentioning that there is evidence that the utility of symplectic methods in convex geometry may extend far beyond Mahler’s conjecture. In a fascinating paper, Akopian, Karasev, and Petrov [4] show that a conjectured weak finite subadditivity property of the Hofer–Zehnder capacity would recover the results of Ball on Bang’s plank problem, another major open question from a different area of convex geometry.

A further development hails from the birth of systolic geometry in 1949, when Loewner discovered that

$$\frac{\text{area}(\mathbb{T}^2)}{\text{sys}(\mathbb{T}^2)^2} \geq \frac{\sqrt{3}}{2},$$

where  $\mathbb{T}^2$  is the 2-torus equipped with any Riemannian metric  $g$ , and the *systole*  $\text{sys}$  denotes the least length of a noncontractible loop. Loewner’s inequality appeared in print a few years later, in an article by his student Pu [195], who proved the corresponding inequality for the projective plane  $\mathbb{P}^2$ , with constant  $2/\pi$  on the right-hand side and equality precisely for the round geometry. In the convex geometry setting, Pu’s theorem implies that if  $K$  is a centrally symmetric convex body in  $\mathbb{E}^3$  with sufficiently smooth boundary, then  $\lambda_2(\partial K)/\text{sys}(\partial K)^2 \geq 1/\pi$ , where  $\text{sys}(\partial K)$  is the least length of a closed geodesic; see, for example, [14, p. 649]. Equality holds when  $K = B^3$ , but there are non-symmetric convex bodies  $K$  for which the ratio  $\lambda_2(\partial K)/\text{sys}(\partial K)^2$  is arbitrarily close to  $1/2\sqrt{3} < 1/\pi$ . Inequalities such as Loewner’s and Pu’s are termed *isosystolic inequalities* and can be viewed as reverse forms of standard isoperimetric inequalities. Álvarez Paiva, Balacheff, and Tzanev [15, Conjecture II and p. 968] make the following conjecture.

(ABT). If a reversible optical hypersurface in the cotangent bundle of  $\mathbb{P}^n$  encloses a volume  $V$ , it carries a periodic characteristic whose action is at most  $V^{1/n}/2$ . Moreover, this short characteristic can be chosen so that its projection onto the base manifold is a non-contractible closed geodesic.

(In [15],  $V$  is  $n!$  times the usual symplectic volume.) Reversible optical hypersurfaces are those whose intersections with cotangent spaces are quadratically convex (i.e., all osculating quadratics are ellipsoids) and centered closed hypersurfaces. As is explained in [15], statements such as ABT have implications in the geometry of numbers. According to J. C. Álvarez Paiva (private communication), a stronger “folklore” conjecture may be formulated in terms of Finsler geometry:

*Finsler metric periodic geodesic conjecture* (FPG). A Finsler metric on  $n$ -dimensional real projective space  $\mathbb{P}^n$  with the same Holmes–Thompson volume as the canonical metric carries a non-contractible periodic geodesic whose length is at most  $\pi$ .

By [15, p. 968], ABT is the special case of FPG that applies to reversible Finsler metrics  $F$  (those for which  $F(-v) = F(v)$  for all tangent vectors  $v$ ). This fact and [15, Theorem VII] means that

$$\text{FPG} \Rightarrow \text{ABT} \Rightarrow \text{MAH2}.$$

No relation between FPG and MAH1 is known. In the same paper in which they prove the local version of VIT mentioned above, Abbondandolo and Benedetti [1] also establish a local version of FPG. For  $n = 2$ , ABT was proved by Ivanov [116], who thereby extended Pu’s inequality. In the language of contact geometry, an odd-dimensional counterpart of symplectic geometry, both ABT and VIT can be viewed as saying that under an appropriate convexity assumption, “Zoll contact forms, i.e., those such that all the orbits of the induced Reeb flow are periodic with the same period, have minimal contact systolic volume,” where the systolic volume is the general term for quantities such as those on the left-hand sides of Loewner’s and Pu’s inequalities. Despite this commonality, no direct implication between ABT and VIT seems to be known, nor has a natural conjecture been formulated that implies both of them.

### **Problem 9.3**

This problem, also stated by Schneider [216, (9.57), p. 515], has now been completely solved by E. Milman and Yehudayoff [178]. Before describing their admirable results, we summarize a couple of earlier developments.

Dafnis and Paouris [65] introduced the *normalized  $i$ th affine quermassintegral*, which can be defined for  $K \in \mathcal{K}_0^n$  and  $1 \leq i \leq n - 1$  by

$$\Phi_{[i]}(K) = \lambda_n(K)^{-1/n} \left( \frac{\kappa_n}{\kappa_i} \right)^{1/i} \Phi_{n-i}(K)^{1/i},$$

and noted that Problem 9.3 is equivalent to asking whether

$$\Phi_{[i]}(K) \geq \Phi_{[i]}(B^n),$$



with equality if and only if  $K$  is an ellipsoid. They asked whether there exists a universal constant  $c_1 > 0$  such that

$$\Phi_{[i]}(K) \geq c_1 \sqrt{n/i},$$

for  $1 \leq i \leq n - 1$ . This was proved by Paouris and Pivovarov [191, Theorem 5.1], providing an affirmative answer to Problem 9.3 in the asymptotic sense, but this is now a consequence of [178, Theorem 1.2].

Zou and Xiong [228] prove the inequality

$$\kappa_n^{(n-i)/n} \lambda_n(P_i K)^{i/n} \leq \Phi_{n-i}(K),$$

for  $1 \leq i \leq n - 1$ , where  $P_i K$ ,  $1 \leq i \leq n$ , is an ellipsoid they call the *ith projection mean ellipsoid*, defined via a constrained optimization problem. Equality holds when  $2 \leq i \leq n - 1$  if and only if  $K$  is an ellipsoid and when  $i = 1$  if and only if there is a  $\phi \in SL(n)$  such that  $\phi K$  has constant width. They note that  $P_n K$  is the so-called Petty ellipsoid, i.e., the  $L_1$  John ellipsoid (see p. 381). It is known that  $\lambda_n(K) \geq \lambda_n(P_n K)$  but the authors show that it is not generally true that  $\lambda_n(K) \geq \lambda_n(P_i K)$  when  $1 \leq i \leq n - 1$ .

As was mentioned above, Milman and Yehudayoff [178] have now solved Problem 9.3. Moreover, their results are stronger in two ways. Firstly, they prove in [178, Theorems 1.2 and 5.1] that for  $1 \leq i \leq n - 1$  and  $u \in S^{n-1}$ , we have

$$\Phi_{n-i}(K) \geq \Phi_{n-i}(S_u K),$$

with equality for all  $u \in S^{n-1}$  if and only if  $K$  is an ellipsoid, where  $S_u K$  is the Steiner symmetral of  $K$  in the direction  $u$ . This surprising result is new even for  $i = n - 1$ . (Note that in [178] the affine quermassintegrals are indexed differently to the usual custom.) This yields the desired inequality via the usual method of successive Steiner symmetrizations (see Lemma 9.2.3). Secondly, they show in [178, Theorem 1.3] that among all convex bodies in  $\mathbb{E}^n$  of a given volume, ellipsoids are the only local minimizers of  $\Phi_{n-i}$  with respect to the Hausdorff metric.

In addition, Milman and Yehudayoff [178, Section 8.2] also find a short proof of the Petty projection inequality, Theorem 9.2.9, that avoids use of the Busemann–Petty centroid inequality, Corollary 9.2.7; instead, their proof is analogous to the one presented for Theorem 9.2.11, the Blaschke–Santaló inequality for centered convex bodies, which was found by Meyer and Pajor [605book, Lemma 1]. In particular, they prove that for  $u \in S^{n-1}$  and  $y \in u^\perp$ , we have

$$\lambda_1((\Pi^* K) \cap (l_u + y)) \leq \lambda_1((\Pi^* S_u K) \cap (l_u + y)).$$

Interestingly, this can be expressed as  $X_u(\Pi^* K) \leq X_u(\Pi^* S_u K)$ , where  $X_u$  denotes the X-ray in the direction  $u$ , and similarly, (9.2) on p. 362 is equivalent to  $X_{u^\perp} K^* \leq X_{u^\perp} S_u^* K$ , where  $X_{u^\perp}$  signifies the  $(n - 1)$ -dimensional X-ray parallel to  $u^\perp$ . Yet another formulation of the two results would be  $S_u(\Pi^* K) \subset S_u(\Pi^* S_u K)$  and  $S_{u^\perp} K^* \subset S_{u^\perp} S_u^* K$ , where  $S_{u^\perp}$  denotes Schwarz symmetrization parallel to  $u^\perp$ .

#### **Problem 9.4**

Milman and Yehudayoff [178, Theorem 8.2] provide an affirmative answer when  $i = 1$ , thus establishing for the first time a sharp Blaschke–Santaló inequality for compact sets. In [178,

Section 8.1], they comment on the possibility of similarly extending their results for convex bodies to compact sets when  $1 < i \leq n - 1$ , but leave these cases open.

**Problem 9.5**

The conjectured inequality is also stated by Schneider [216, (9.56), p. 515]. As is pointed out in Note 9.6, the special case  $i = 0$  reduces to Problem 9.3. Apart from this, the problem is still wide open.

Milman and Yehudayoff [178] introduce the  $L_p$ -moment affine quermassintegrals, defined (in our notation) for  $K \in \mathcal{K}_0^n$ ,  $0 \leq i \leq n - 1$ , and  $0 > p \geq -n$ , by

$$\Phi_{n-i,p}(K) = \frac{\kappa_n}{\kappa_i} \left( \int_{\mathcal{G}(n,i)} \lambda_i(K|S)^p dS \right)^{1/p},$$

when  $1 \leq i \leq n - 1$ , and  $\Phi_{n,p}(K) = \kappa_n$ . Thus  $\Phi_{n-i,-n}(K) = \Phi_{n-i}(K)$  is the usual affine quermassintegral of  $K$ . (The case when  $p = 0$  is also considered in [178, Section 7.2], where it is shown to play a role in an averaged Loomis–Whitney inequality.) They conjecture that for  $0 \leq i \leq j \leq n - 1$  and  $0 > p \geq -n$ , we have

$$(36) \quad \kappa_n^j \Phi_{i,p}(K)^{n-j} \leq \kappa_n^i \Phi_{j,p}(K)^{n-i},$$

with equality for  $p > -n$  if and only if  $K$  is a ball. The case  $p = -n$  is Problem 9.5, where equality is conjectured to hold if and only if  $K$  is an ellipsoid. In [178, Theorem 1.5], (36) is proved when  $0 > p \geq i - n$ , the other cases remaining open.

It is interesting that with one exception, the situation is exactly mirrored in the dual case. Gardner [86, (41), p. 379] introduced what might now be called *dual*  $L_p$ -moment affine quermassintegrals, defined for bounded Borel sets  $C$ ,  $0 \leq i \leq n - 1$ , and  $0 < p \leq n$ , by

$$\tilde{\Phi}_{n-i,p}(C) = \frac{\kappa_n}{\kappa_i} \left( \int_{\mathcal{G}(n,i)} \lambda_i(C \cap S)^p dS \right)^{1/p},$$

when  $1 \leq i \leq n - 1$ , and  $\tilde{\Phi}_{n,p}(C) = \kappa_n$ , and (without stating it as a conjecture) considered the possibility that for  $0 \leq i \leq j \leq n - 1$  and  $0 < p \leq n$ , we have

$$(37) \quad \kappa_n^j \tilde{\Phi}_{i,p}(C)^{n-j} \geq \kappa_n^i \tilde{\Phi}_{j,p}(C)^{n-i}.$$

In [86, Theorem 7.4], (37) is proved, with precise equality conditions, when  $0 < p \leq n - i$ , which are just the values corresponding to those in [178, Theorem 1.5]. The exception mentioned above is that in [86, Theorem 7.7], (37) is actually proved to be false when  $1 \leq i \leq n - 2$ ,  $j = n - 1$ ,  $p = n$ , and  $C$  is a centered convex body that is not an ellipsoid. However, for sections, unlike projections, the case  $j = n - 1$  is different in an essential way; it corresponds to the case  $i = 1$  of [86, Theorem 7.4], where the equality condition indicates this difference. It remains quite possible, then, that (37) is true when  $j \leq n - 2$ .

**Problem 9.6**

Dafnis and Paouris [65] (see the report for Problem 9.3) also asked if there is a universal constant  $c_2 > 0$  such that

$$\Phi_{[i]}(K) \leq c_2 \sqrt{n/i},$$

for  $1 \leq i \leq n - 1$ , and showed that this inequality holds with an extra factor of  $\log n$  on the right-hand side. It has been verified for some classes of random polytopes by Chasapis and Skarmogiannis [58].

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